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Sequencing heterogeneous orders in Bucket Brigade order-picking lines

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ABSTRACT

Bucket Brigade (BB) is a common approach for dynamic work-sharing in order-picking lines. Differently from typical analysis, we assume heterogeneous orders, which creates blockages and reduced efficiency. The problem is how to sequence entering orders with the aim of minimizing this potential inefficiency. The proposed framework models order-picking lines with workload distributed along the picking aisle according to the number of items to be picked in each pick face. We propose a measure for quantifying the generated blockage inefficiency (BI) as a proxy for the makespan. As the BI depends on the sequence of orders, several strategies are proposed to identify sequences with no-blockage or with minimal BI. We provide several practical sequencing policies. Sequencing based on no-blockage notions and steady-state hand-off positions is proved useful, and no-blockage is implied by first-order distributional dominance sequencing. Traveling salesman problem and Hamiltonian path modeling is proposed as an exact computational method of item-specific sequences with minimal or no blockage in a strong sense. A simple policy ensures low BI for large sets of orders, for which we show an asymptotic result: the BI of any efficient sequence approaches zero in the limit as the sequence length tends to infinity. In general, sequencing orders is a practically relevant and effective managerial strategy, as it typically substantially reduces the BI and often eliminates it entirely.

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1. Introduction

Modern environments of order-picking apply work-sharing methods, whereby multiple cross-trained workers share the sequential execution of multiple operations on a given order to complete a product or service. Such environments become quite complex when demand is customer specific and orders are heterogeneous, as the system control becomes more challenging and operational performance may deteriorate. Possibly the most commonly analyzed work-sharing approach in serial lines is the Bucket Brigade (BB) proposed by Bartholdi and Eisenstein (1996). According to BB in order-picking lines, whenever the last downstream worker in the picking aisle completes an order, the worker moves back and takes over the order from the immediate upstream worker, this upstream worker continues similarly with the next upstream worker, and the procedure continues until the first upstream worker starts processing a new order.

In general, serial lines suffer from throughput loss due to halting, starvation and blockage. The former two issues mostly do not exist in a fully cross-trained BB system, whereas the latter occurs when workers are not allowed to overtake each other. Most of the relevant literature, as well as the current article, assume that overtaking is not allowed. This stems from the fact that overtaking is very difficult to apply in real-world order-picking in forward storage areas, due to technological reasons and cost of footprint. Bartholdi and Eisenstein (1996) suggested conditions ensuring a self-balancing BB line under the assumptions of uniform orders, full cross-training of the workers, no restriction on the hand-off point (the point in which a job is handed from one worker to another), and a completely deterministic nature of the system. They show that when the workers are arranged from slowest to fastest, the line does not suffer from blockages, thus achieving the maximal theoretical throughput rate. Additionally, they show that under the above assumptions the hand-off positions converge to a constant steady-state, consequently each worker specializes on executing their own work segment for each order.

One of the common assumptions in the BB literature is that of homogeneous orders, namely identical orders with planned workload uniformly distributed along the line/aisle. Homogeneous orders may include the possibility of stochastic workload, in that the planned workload in each pick face along the aisle has the same probabilistic distribution, but its realization may vary across orders. In this article we follow Hong *et al.* (2016) and Fibrianto and Hong (2019) by relaxing the assumption of homogeneous orders and considering heterogeneous orders, i.e., orders that differ from one

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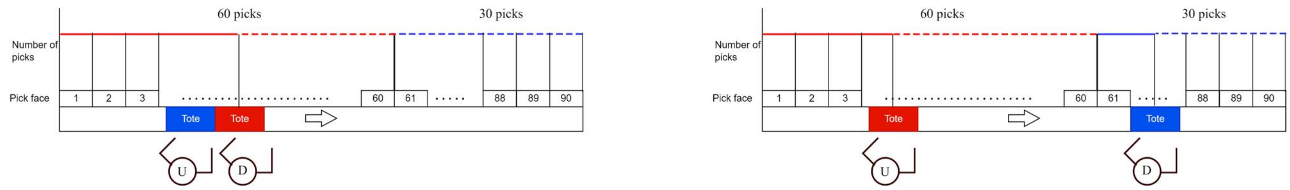


(a) sequence with blockages (b) efficient sequencing

Figure 1. Sequencing heterogeneous orders.

another in their planned total workload or its distribution along the line. Such heterogeneity generates potential blockages that may be eliminated via sequencing. The following small example demonstrates the importance of order sequencing. Say there is a picking aisle with 90 pick faces and two orders to pick. Order 1 contains 60 items to be picked from pick faces 1 to 60, and order 2 contains 30 items to be picked from pick faces 61–90. The pick time is one time unit per pick. As demonstrated in Figure 1, two pickers are working in the aisle under a BB regime. We assume that the walking time between picks is negligible and that the pickers work at the same work rate. Consider a sequence in which order 1 (shown in red) precedes order 2 (shown in blue). Solid lines depict the planned workload that was already accomplished, and dash lines the remaining planned workload. In this case, as seen in Figure 1(a), the downstream worker (D) picks order 1, while the upstream worker (U) is blocked. After 60 time units, the downstream worker completes picking order 1, and continues to the end of the line to submit the order. Then a hand-off occurs at pick face 61, where the downstream worker starts picking order 2, while the upstream worker goes to the start of the line to take the next order (if any). The total pick time of the two orders (the total time of the two cycles) equals 90 time units. Now consider the opposite sequence of the orders, namely order 2 precedes 1. Figure 1(b) shows both workers picking simultaneously, the downstream worker with order 2 and the upstream worker with order 1. After 30 time units, the downstream worker completes picking order 2 and returns to the upstream worker. A hand-off occurs at pick face 31, where the downstream worker starts picking order 1, while the upstream worker goes to the start of the line to take the next order (if any). The total pick time of the two orders is now equal to 60 time units, 33% lower than in the previous sequence. The example suggests that even when the two workers have the same, constant work rate, the different planned workloads may lead to potential blockages, which would be significantly affected by the arrival sequence of the various orders into the line.

In general, blockages impair the line's throughput, due to the partial utilization of the blocked workers' potential work rates. Therefore, although a main motivator for implementing a BB line is its successful ability to solve blockage problems stemming from random variability in the process when orders are homogeneous, BB lines are far from adequate in handling blockage problems when the process involves systematic, planned for heterogeneity across the orders. Missing from the literature is a general understanding of simple and efficient operational policies for BB lines with heterogeneous orders. This is the main contribution of this article.

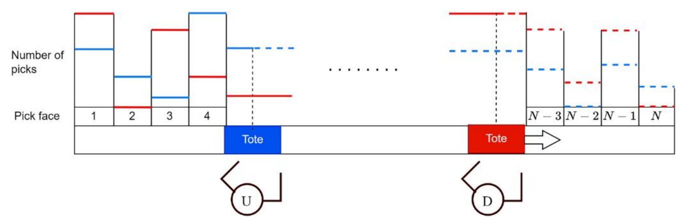


Figure 2. General example illustrating an order-picking process with heterogeneous orders.

We propose to investigate BB lines with given heterogeneous orders, further controlled using order sequencing. Our proposed modeling approach fits common real-world order-picking processes in forward storage areas, where each order may contain a different set of Stock Keeping Units (SKUs) / items resulting in different total picking requirements, or different workload distributions. As shown in Figure 2, two pickers work in an aisle and the workload is distributed according to the number of items to be picked at each pick face.

Order heterogeneity generates potential blockages in BB order-picking lines, as discussed above. The amount of blockage is affected by the particular set of orders and the exact sequence in which they enter the line. Two common operational approaches to reducing blockages include order batching and sequencing. The former determines the set of orders and is applicable for order-picking, whereas the latter determines the order sequence and is applicable in general. In this article we concentrate on sequencing as the operational strategy given the set of orders, and our contribution is to show that this is very effective in handling the blockage problems. To this end, we propose a measure of Blockage Inefficiency (BI), defined for a given sequence of orders as the work capacity loss due to blockage divided by the total workload for all orders. As this BI might be high or low depending on the chosen sequence of orders, we propose to solve a static sequencing problem for finding efficient sequences that minimize the BI. We show that this objective may be seen as a proxy for the minimization of the production makespan (MS).

We demonstrate with simple examples the potentially severe inefficiency that we might obtain, depending on the work rates and the sequence of orders. Nevertheless, our general conclusion is that the sequencing problem we formulate is practically very relevant, as it typically substantially reduces this inefficiency and often eliminates it entirely.

This is relevant when the number of orders is either small or large.

To establish the above general conclusion, we propose several sequencing policies. As preliminary analysis, we provide necessary and sufficient conditions for no-blockage and for blockage during a cycle using a cumulative workload difference function that also takes into account the work rates.

Our first result concerns non-uniform identical orders, for which we show that no-blockage in any cycle is guaranteed as long as the workers are positioned slowest to fastest. This analysis leads to a generalization of Bartholdi and Eisenstein's (1996) steady-state hand-off positions to orders with non-uniform distributions.

This approach is extended in several directions. We first propose to check easily verifiable conditions on the given set of orders, thus ensuring no-blockage in general. The analysis is based on particular, nested notions of no-blockage. The first, most demanding notion involves a set of order types with *universal no-blockage*, in the sense that no blockage occurs for any sequence of order types selected from the set. Moreover, this set of orders is *maximal* in the sense that including in the set any additional order type from outside the set necessarily violates the universality requirement. We provide an algorithm for determining the inclusion in such a set of order types, which therefore ensures that any sequence has no-blockage.

We then propose three sequencing policies. The first sequencing policy concerns sorting the orders according to non-increasing weighted average of steady-state hand-off positions. Here we consider orders with identical total workload, but possibly different distributions. For such orders we show that this sequencing policy can ensure no-blockage, and demonstrate numerically that the policy effectively decreases the BI. The policy is also compared numerically to the other two sequencing policies for general orders.

The second sequencing policy is based on a second, less demanding notion of no-blockage. It involves a sequence of orders with *strong no-blockage*, in the sense that no blockage would occur when each consecutive pair of orders in the sequence were processed alone. We show that first-order distributional dominance is sufficient for strong no-blockage, thus there is a wide opportunity for strong no-blockage. We also propose an extension to general orders by solving a Travelling Salesman Problem (TSP) formulation based on strong no-blockage. In particular, Hamiltonian paths in the zero cost subgraph are strong no-blockage sequences.

The third and last sequencing policy generates low BI when the number of orders to be sequenced is relatively large. This policy involves a sequence in which groups of identical orders are ordered lexicographically according to increasing total workload and then decreasing steady-state hand-off position. We use the sequencing policy for piecewise linear orders, which are relevant e.g. in order-picking processes (as in Figure 2). For such orders we demonstrate substantial reductions in the BI, and also that for piecewise linear order types with any given approximation level and slowest to fastest assignment of the workers, the BI of any efficient sequence approaches zero in the limit as the sequence length tends to infinity. This result is important also because piecewise linear order types are a good approximation for the domain of all order types when the number of break points is large. The analysis is supplemented by a numerical experimentation which demonstrates how the above three sequencing policies reduce the BI for general orders.

In the remainder of the Introduction we review the relevant literature on work-sharing systems and BB in particular. Systems that adopt work-sharing differ by the level of worker cross-training, ranging from full cross-training where all workers are capable of performing the whole task, to partial cross-training where there is only some overlapping between the capabilities of the workers. On the operational side, Ostolaza *et al.* (1990) use the term Dynamic Line Balancing (DLB) to describe predefined tasks that are shifted dynamically between adjacent stations/workers based on the state of the system. Hopp *et al.* (2004) investigated these issues in the context of achieving a low ratio of work-in-process (WIP) to throughput. Besides the level of cross-training, they investigated the line topology by studying D-Skilled Chaining (DSC), which was initially coined by Jordan *et al.* (2004), and Cherry Picking (CP). In CP, only the bottleneck worker is assisted by the other workers, while the cross-training under DSC is symmetric among the workers, as each worker can help the adjacent upstream/downstream workers. The literature has also addressed other variations of work-sharing systems, such as preemption by which a task may be split between workers (McClain *et al.*, 2000), and processes in which machines are involved (Zavadlav *et al.*, 1996). We note that the scheduling literature typically does not address these issues, due to the assumption taken in this literature that processing times are not affected by item sequencing.

Following Bartholdi and Eisenstein (1996), several extensions have been suggested (see Bratcu and Dolgui (2005) for a review). In the deterministic domain with homogeneous items, the BB system dynamics for two and three workers was analyzed in Bartholdi *et al.* (1999). The chaotic behavior of the hand-off point when the convergence condition does not hold was studied in Bartholdi *et al.* (2009). Armbruster and Gel (2006) relaxed the dominance assumption, and studied a two-worker line in which one worker may be faster than his neighbor in some part of the line, and slower in the other part. They provided insights and operation principles for various scenarios. Gurumoorthy *et al.* (2009) analyzed the dynamics of a line with two workers, each having an arbitrary, constant speed at each station. When considering discrete workstations, the results of the basic continuous model approximately hold for large number of stations, however, a different analysis is needed when the number of stations is small. This issue was considered in Lim and Yang (2009), who found conditions under which two- and three-station lines maximize throughput for a given number of stations. Bartholdi *et al.* (2006) applied the BB principles in an in-tree network of sub-assembly lines. Bratcu and Dolgui (2009) and Lim (2011) relaxed the common assumption of infinite worker walk back speed, where the latter proposed a cellular configuration model. Lim and Wu (2013) studied a U-line BB system with discrete stations. In another direction, Armbruster *et al.* (2007) and Montano *et al.* (2007) studied the effect of learning in BB systems, where the latter suggested an alternative control rule named modified-work-sharing. Some industrial case studies of BB have been studied in Bartholdi and Eisenstein (2005), Lim (2005) and De Carlo *et al.* (2013).

In the stochastic domain with homogeneous items, Bartholdi *et al.* (2001) studied the performance of BB under the assumption of stochastic processing times, and showed that the throughput rate converges to its optimal value as the number of stations increases. Buzacott (2002) considered four-station and two-worker stochastic lines with or without preemption and derived the optimal policy, which modifies the non-preemptive BB rule. Bratcu and Dolgui (2009) studied a BB system with stochastic performance rates via simulations, assuming that both working and walk back speeds are normally distributed. Hong (2014) derived an analytic expression for the two-worker blocking congestion in a circular-passage system under the assumptions of constant worker speeds and probabilistic picks. In a later paper, Hong *et al.* (2015) provided a closed-form expression to the level of blocking for two extreme walk speed cases. Bukchin *et al.* (2018) studied three variants of BB production lines under the assumption of stochastic worker speeds: the traditional BB line, BB with Overtaking allowed (BBO), and a benchmark system of parallel workers. They showed that BB lines may perform better than a comparable system with parallel workers, and that the best configuration is BBO. Additionally, they showed that slowest to fastest is not always optimal when speeds are stochastic. Wang *et al.* (2022) study a BB system with discrete work stations where the time duration for each worker to process an item at a station is exponentially distributed with a rate that depends on the station's work content and the worker's work speed. The general conclusion that BB lines are immune to stochastic workloads even without sequencing is a direct consequence of the assumption of homogeneous orders. In contrast, our contribution is to show that sequencing policies are very much required when orders are heterogeneous.

Hong *et al.* (2016) and Fibrianto and Hong (2019) considered heterogeneous items in deterministic environments. They developed a batching and sequencing Mixed-Integer Programming (MIP) formulation to reduce blocking delays in BB order-picking lines with work-content-dependent picking times. Integrating a "rolling horizon" implementation of this Mixed-Integer Programming (MIP), they considered at each step a small number of orders within a simulation with stochastic picking times and compared the results with a random policy. Our contribution compared with these papers is to offer simple and optimal sequencing policies that typically eliminate the BI almost entirely, and this is shown analytically and numerically. Such policies may be used as substitutes or in addition to batching operations.

The rest of this article is organized as follows. In Section 2 we present the model and propose our measure of BI for a BB line with heterogeneous orders. In Section 3 we propose several sequencing policies for reducing the BI, and provide an evaluation of these policies using formal results, examples and numerical studies. Section 4 discusses generalizations and concludes. A preliminary analysis of no-blockage and blockage sequences is included in Appendix A, and all proofs are collected in Appendix B.

2. Model

We study a BB line (Bartholdi and Eisenstein, 1996) for coordinating the efforts of several workers along a forward storage order-picking aisle. The protocol of BB includes forward (downstream) and backward (upstream) movements of each worker along the line. During a forward movement, the worker is involved in picking an order. Moving forward ends for the most downstream worker upon reaching the end of the line with a finished order, and ends for any other worker upon meeting their immediate downstream worker who is moving backward. When moving backward, the worker does not hold any order. Moving backward ends for the most upstream worker upon reaching the start of the line, and ends for any other worker upon meeting their immediate upstream worker who is moving forward. When such a meeting occurs, the downstream worker takes the order from the upstream worker and starts a forward movement to continue processing it, and the upstream worker starts a backward movement to take a different order from an upstream worker or from the start of the line. Therefore, the completion of an order by the most downstream worker initiates a sequence of such meetings, which are together called a hand-off event.

The work rate of each worker $k = 1, \dots, K$, i.e., the amount of work that the worker can process in a unit of time, is fixed at $r_k > 0$. In line with most of the literature, we assume that the time required for any worker to return upstream is insignificant compared with the time required to work downstream, thus all workers hand-off simultaneously and the duration of any hand-off event may be ignored. The sequence of the workers along the line does not change over time, as overtaking is not allowed. As a result, each worker either proceeds at their own work rate or at a reduced pace when blocked by the next downstream worker. To be consistent with our focus on blockage (not starvation) problems, we assume a continuous line, namely that a hand-off can occur at any position along the line, and without loss of generality, identify the line with the interval [0, 1]. At time 0, a given set of heterogeneous orders is ready to be processed at the start of the line. Order $j = 1, 2, \dots, J$ is identified by an order type, i.e., a bounded workload density function $w_j : [0, 1] \to \mathbb{R}_+$, for which the corresponding cumulative workload distribution function, $W_j(x)$, is assumed to be well defined for any $x \in [0, 1]$ as the integral of $w_j(x)$ over [0, x]. Differently from probability distributions, the total workload $W_j(1)$ may be above, equal or below one. The permutation sequence of the orders, denoted by $s = (s_1, \dots, s_J)$, where s_j is the order at position $j = 1, \dots, J$ in the sequence, is given before production starts. We aim at determining the appropriate sequence of orders s in order to minimize the blockage inefficiency (formal definition provided below), thus a set of orders $W = \{W_j, j = 1, ..., J\}$ together with a vector of work rates $r = (r_k, k = 1, ..., K)$ will be referred to as a sequencing problem (W, r).

Fixing the sequence of the workers as $1 \to 2 \to ... \to K$, we mostly assume a slowest-to-fastest configuration, i.e., $r_1 \le ... \le r_K$, and for $k = 1, ..., K-1$ denote the work rate ratio of worker k to worker $k+1$ by $\bar{r}_k = \frac{r_k}{r_{k+1}}$. Given a sequencing problem (W, r), for any sequence s, the nth hand-off state of the system, $x_n(W, r, s) = (x_{n,k}(W, r, s), k = 1, ..., K)$ for $0 \le x_{n,k}(W, r, s) \le 1$, defines the position along the line of each worker k just before the nth hand-off for $n = 1, 2, ..., J-1$. At these positions, worker $k = \max\{1, n+K-J\}, ..., K$ is with order s_{n+K-k}, and the most downstream worker K has reached the end of the line, i.e., $x_{n,K} = 1$ for $n = 0, 1, ..., J$. We assume $x_{0,k}(W, r, s) = 0$ for all worker $k = 1, ..., K-1$ and any sequence s. Since worker K cannot be blocked and always reaches the end of the line, the cycle time, i.e., the time elapsed between hand-off $n-1$ and hand-off n, is always

$$CT_n(W, r, s) = \frac{W_{s_n}(1) - W_{s_n}(x_{n-1,K-1}(W, r, s))}{r_K}. \tag{2.1}$$

Accordingly, we next define the system MS.

Definition 2.1. The system MS for sequence s is the total time to complete processing J jobs,

$$MS(W, r, s) = \sum_{n=1}^{J} CT_n(W, r, s). \tag{2.2}$$

Blockage may occur over any partial interval of the line during any cycle $n < J$, depending on the work rates and the type of the order held by each worker. Such blockages create inefficiency, due to the reduced pace of any upstream worker, which in turn generates longer cycle times, and consequently a higher MS. The maximal amount of work potentially accomplished by any worker is achieved when each worker proceeds in their own work rate, i.e., when there is no blockage. We next propose a measure to quantify this blockage inefficiency. Minimizing this measure is then shown to be a proxy for minimizing the MS.

Definition 2.2. The *makespan work capacity* for sequence s is the maximal amount of work potentially accomplished by the workers during the makespan, i.e. if there were no blockages,

$$MC(W, r, s) = \sum_{n=1}^{J} CT_n(W, r, s) \left(\sum_{k=\max\{1, n+K-J\}}^{K} r_k \right). \tag{2.3}$$

Note that the most downstream worker K is busy in all cycles, whereas any upstream worker $k = 1, ..., K-1$ is busy in all but the cycles $n > J - K + k$. Now, defining the total workload of all orders by

$$TW(W) = \sum_{j=1}^{J} W_j(1), \tag{2.4}$$

the difference

$$BL(W, r, s) \equiv MC(W, r, s) - TW(W) \tag{2.5}$$

will be referred to as the *work capacity loss due to blockage* for sequence s. Using these notions, we may define our measure of inefficiency.

Definition 2.3. The BI for sequence s is

$$BI(W, r, s) = \frac{BL(W, r, s)}{TW(W)} = \frac{MC(W, r, s) - TW(W)}{TW(W)}. \tag{2.6}$$

We are interested in sequences s that guarantee low values of our inefficiency measure, $BI(W, r, s)$. It is useful to compare the BI to the MS Inefficiency (MSI), the ratio given by the difference between the MS and the theoretical lower bound for the MS divided by this lower bound, defined by

$$\begin{aligned} MSI(W, r, s) &= \frac{MS(W, r, s)}{TW(W) / \sum_{k=1}^{K} r_k} - 1 \\ &= \frac{\sum_{n=1}^{J} CT_n(W, r, s)}{TW(W) / \sum_{k=1}^{K} r_k} - 1, \end{aligned}$$

while noting that

$$\begin{aligned} BI(W, r, s) &= \frac{\sum_{n=1}^{J} CT_n(W, r, s) \left(\frac{\sum_{k=\max\{1, n+K-J\}}^{K} r_k}{\sum_{k=1}^{K} r_k} \right)}{TW(W) / \sum_{k=1}^{K} r_k} - 1 \\ &\le MSI(W, r, s). \end{aligned}$$

For $J \ge K$, excluding the sequence tail consisting of the last $K-1$ cycles in which not all workers are busy, the components of the BI and the MSI are identical. This tail is negligible for practical sized sequences where $J \gg K$. Therefore, minimizing the BI over all sequences is a proxy to minimizing the MS over all sequences. This is formalized by the following proposition.

Proposition 2.1. *For any sequencing problem (W, r) with a large number of orders, any sequence that minimizes the BI approximately achieves minimum MS.*

To compute the BI for a sequence s we apply for each cycle a recursive procedure from worker $K-1$, the last worker that can be blocked, back to worker 1. After calculating the contribution of each worker k to the BI, we modify the order this worker processes by adding virtual work exactly so that this worker is no longer blocked by the next downstream worker. Then we proceed with this modified order to calculate the contribution of worker $k-1$ to the BI. More details are given in Appendix A.

Blockage inefficiency has a simple tight upper bound: the ratio of work rates, $\frac{\sum_{k=1}^{K-1} r_k}{r_K}$. This is stated in Proposition 2.2. The proof of the proposition shows that high inefficiency occurs in particular when an order with large total workload is followed by an order with a small total workload.

Proposition 2.2. *For any sequencing problem (W, r) and any sequence s, an upper bound for $BI(W, r, s)$ is $\frac{\sum_{k=1}^{K-1} r_k}{r_K}$, and this bound is tight.*

A higher work rate ratio $\frac{\sum_{k=1}^{K-1} r_k}{r_K}$, which equals in particular $\bar{r}_1$ when $K = 2$, may therefore generate higher inefficiency. Nevertheless, due to order heterogeneity, blockages in early cycles may alleviate blockages in later cycles. Specifically, $BI(W, r, s)$ is not always increasing with the work rate ratio $\frac{\sum_{k=1}^{K-1} r_k}{r_K}$, and may even decrease from a positive value to zero. In particular, there exists a set of orders W, a sequence s and r^1, r^2, where $K = 2$, with $\bar{r}_1^1 < \bar{r}_1^2$ such that $BI(W, r^1, s) > BI(W, r^2, s) = 0$.

A main subject of interest in our analysis will be sequences with no-blockage.

Definition 2.4. A sequence s has *no-blockage* for (W, r) if $BI(W, r, s) = 0$, i.e., for any cycle n, no worker is blocked by the subsequent downstream worker throughout the cycle.

3. Sequencing policies

Our approach leads to several sequencing policies, described and analyzed in the subsections below.

3.1. Non-uniform identical orders

The simplest case in our analysis is where $W = (\bar{W}, ..., \bar{W}) = \bar{W}$, i.e., all orders are identical, say with type $\bar{W}$, but the workload may not be distributed uniformly across the line. Clearly, there is no meaning to sequencing of identical orders. However, given that we analyze the effect of order heterogeneity, it is useful to ask as a starting point whether identical orders with non-uniform distributions create potential blockages. We show that such issues do not arise. Intuitively, for this case, since at any position x the instantaneous amount of work required by each worker is the same, if the workers are ordered slowest-to-fastest, blockage cannot occur at any position x. This intuition is confirmed by the following no-blockage result.

Proposition 3.1. *For any sequencing problem $(\bar{W}, r)$ with work rate ratio $\bar{r}_k \leq 1$ for $k = 1, ..., K-1$, any sequence has no-blockage.*

In steady-state, the hand-off position remains unchanged across cycles. This leads us to propose the following definition, which generalizes Bartholdi and Eisenstein (1996)'s definition for uniform, identical orders.

Definition 3.1. The *steady-state hand-off position* of type $\bar{W}$ is $x^*(\bar{W}) = [x_1^*(\bar{W}), ..., x_K^*(\bar{W})]$ given for all $k = 1, ..., K$ by[1]

$$x_k^*(\bar{W}) = \max\left\{x_k | \bar{W}(x_k) = \bar{W}(1) \frac{\sum_{k'=1}^{k} r_{k'}}{\sum_{k=1}^{K} r_k}\right\}.$$

The following proposition shows that the no-blockage property of non-uniform identical order sequences guarantees the convergence to steady-state.

Proposition 3.2. *For any sequencing problem $(\bar{W}, r)$ with work rate ratio $\bar{r}_k < 1$ for all $k = 1, ..., K-1$ and any sequence s, hand-off position n, $x_n(\bar{W}, r, s)$, converges to the steady-state hand-off position $x^*(\bar{W})$ as the number of orders J approaches infinity, and the convergence rate is exponential.*

Proposition 3.2 leads to the following important insight: self-balancing is achieved so that each worker specializes on repeatedly executing a fixed work segment. This implication is due to the orders being identical rather than being uniform.

3.2. Sequence-independent no-blockage

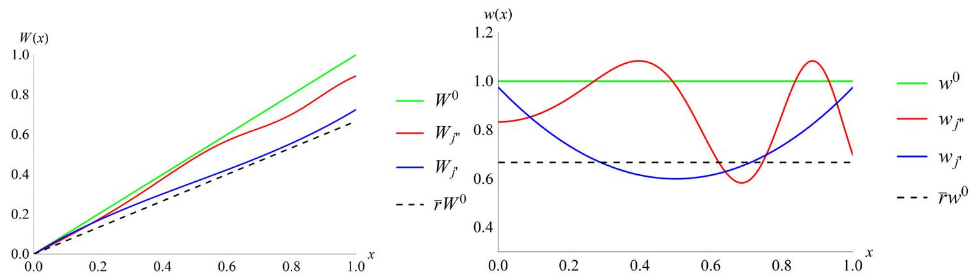
Two of the sequencing strategies we propose are based on particular, nested notions of no-blockage. The following approach is based on the first, most demanding no-blockage notion, which therefore has the advantage of ensuring no-blockage in a very wide set of scenarios. Recall the definition of order types in the second paragraph of Section 2. A consequence of our analysis is that if a system designer can affect which order types may enter the line, for example by batching orders, the approach described here suggests a way to ensure efficiency without the need to worry about order sequencing. Specifically, we investigate sets of order types for which no blockage occurs in any sequence, which generalizes the case of identical orders studied in the previous subsection. This is formalized in the following definition we propose.

Definition 3.2. Given work rates r, a set of order types satisfies *maximal universal no-blockage* if:

(a) Universality: for any sequence of orders with order types selected from the set, no blockage occurs in any cycle; and
(b) Maximality: including in the set any additional order type from outside the set necessarily violates the universality requirement (a), i.e., there would exist a sequence of orders with order types selected from the set including the additional order type, for which blockage occurs in some cycle.

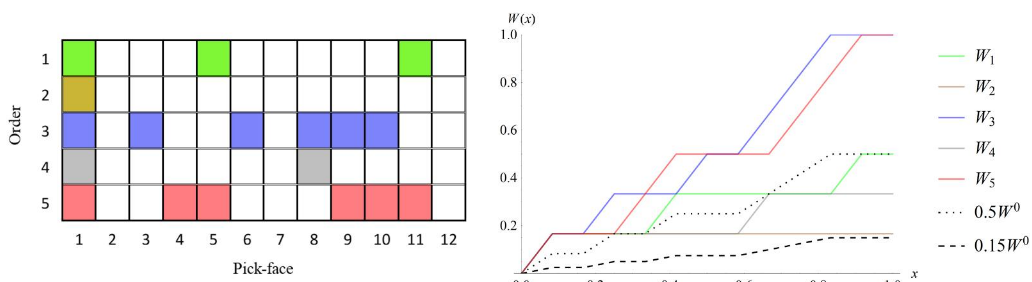
Our next result characterizes sets of order types satisfying maximal universal no-blockage. Given any maximal work rate ratio $\bar{r} \equiv \max_{k=1,...,K-1} \bar{r}_k$ and order type W^0, denote by $\mathcal{H}_{\bar{r}, W^0}$ the set of all order types $\hat{W}$ such that $\hat{W}(x) \in [\bar{r} W^0(x), W^0(x)]$ for all $x \in [0, 1]$. Note that $\mathcal{H}_{\bar{r}, W^0}$ is nonempty if and only if $\bar{r} \leq 1$, i.e., a slowest to fastest configuration of workers. There are infinitely many such sets, and one of them is illustrated in Figure 3 for a linear (uniformly distributed) order type W^0, together with two possible nonlinear order types within this set. As shown in the figure,

[1]The max is relevant when $\bar{W}(x)$ is constant because the workload density is zero.



(a) cumulative workload (b) workload density

Figure 3. Set of order types satisfying maximal universal no blockage.



(a) picking specification (b) cumulative workload

Figure 4. Five-order subset of a maximal universal no-blockage set.

$\mathcal{H}_{\bar{r},W^0}$ is a strict super-set of the set of all order types with bounded workload density functions, i.e., such that $\hat{w}(x) \in \left[\bar{r}w^0(x), w^0(x)\right]$ for all $x \in [0, 1]$.

Proposition 3.3. *For any maximal work rate ratio $\bar{r} \leq 1$ and order type W^0, the set of order types $\mathcal{H}_{\bar{r},W^0}$ satisfies maximal universal no-blockage.*

The simple case described in Subsection 3.1 of non-uniform identical orders is always a subset of some maximal universal no-blockage set $\mathcal{H}_{\bar{r},W^0}$. In particular, when the workers are identical, i.e., $r_1 = \ldots = r_K$, the maximal universal no-blockage set $\mathcal{H}_{\bar{r},W^0}$ includes the single-order type W^0. Additionally, Proposition 3.3 directly implies the following natural monotonicity property of maximal universal no-blockage with respect to work rate ratios.

Corollary 3.1. *Given an order type W^0, work rates $\hat{r}$ with $\max_{k=1,\ldots,K-1} \frac{\hat{r}_k}{\hat{r}_{k+1}} < \bar{r}$ and a set of orders W with $W_j \in \mathcal{H}_{\bar{r},W^0}$ for each j, any s is a no-blockage sequence for $(W, \hat{r})$.*

The following algorithm may be applied in order to check whether a set of orders W is a subset of some maximal universal no blockage set: Defining W^0 as the pointwise maximum of W_j of all orders j in the set W, i.e., let $W^0(x) = \max_{1 \leq j \leq J} W_j(x)$ for each $x \in [0, 1]$, check whether each W_j is in the set $\mathcal{H}_{\bar{r},W^0}$, i.e., check if $W_j(x) \in \left[\bar{r}W^0(x), W^0(x)\right]$ for all $x \in [0, 1]$ and each order j.

Proposition 3.4. *The decision given by the algorithm always identifies correctly whether a set of orders W is a subset of some maximal universal no blockage set.*

Example 3.1. We demonstrate the maximal universal no-blockage approach with a (W, r) example including five orders to be picked from an aisle with 12 pick faces. At a pick face, each order either has one unit workload or none, as depicted in Figure 4(a), with a corresponding cumulative workload distribution function, $W_j(x)$, depicted in Figure 4(b). Applying the algorithm, the pointwise maximum of all orders in Figure 4(b) is identified as $W^0(x) = \max\{W_3(x), W_5(x)\}$. Then, for $\bar{r} = 0.15$, since $W_j(x) \in \left[\bar{r}W^0(x), W^0(x)\right]$ for all $x \in [0, 1]$ and each order j, W is a subset of the maximal universal no-blockage set $\mathcal{H}_{\bar{r},W^0}$, thus any sequence has no blockage. In contrast, for $\bar{r} = 0.5$, only the orders W_3, W_5 are each in $\mathcal{H}_{\bar{r},W^0}$, therefore only these two form a subset of this maximal universal no-blockage set.

In sum, we find that maximal universal no-blockage is easily identified, and expands when the minimal gap between the work rates increases, thus increasing the no-blockage opportunities.

3.3. Steady-state hand-off positions sequencing

General orders with *identical total workload* allow a simple no-blockage sequencing policy we propose based on the steady-state hand-off position introduced in Definition 3.1. As we will show, under identical total workload, blockage opportunities are reduced when the orders are processed with decreasing steady-state hand-off positions, with higher importance placed on downstream workers. We therefore propose the following sequencing policy.

Definition 3.3. Let s^{sshp} be a sequence in which the orders are sorted according to decreasing weighted average, $\sum_{k=1}^{K-1} \alpha_k x_k^*$, for increasing weights $0 < \alpha_1 \le \ldots \le \alpha_{K-1}$, of steady-state hand-off positions.

The sequencing policy s^{sshp} reduces blockage opportunities, because the difference between the work accomplished by any upstream worker and the work they would accomplish were they to move together with the next downstream worker tends to decrease.

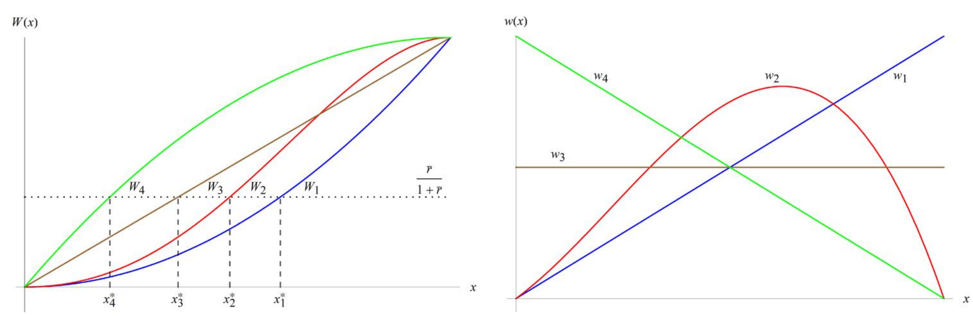
The policy guarantees no blockage under certain conditions, namely a dominance relation between the steady-state hand-off positions, together with the ability to initialize the process, before the first cycle starts, by letting workers $k = 2, \ldots, K$ work up to the relevant steady-state hand-off position of the initial $K-1$ orders in the sequence s^{sshp}:

Proposition 3.5. *Consider a sequencing problem (W, r) where all orders have identical total workload denoted by a, i.e., $W_j(1) = a$ for all j. Suppose that there exists a sequence s^* in which the orders have non-increasing steady-state hand-off positions, i.e., $x_k^*(W_{s_1^*}) \ge \ldots \ge x_k^*(W_{s_J^*})$ for all workers $k = 1, \ldots, K$. Then the same holds for any s^{sshp} sequence. Furthermore, if the starting position of each worker $k = 2, \ldots, K$ in the first cycle is the* (k–1)th *component of the steady-state hand-off position of the* $(K+1-k)$th *order in a sequence s^{sshp}, i.e., $x_{0,k}(W, r, s^{\text{sshp}}) = x_{k-1}^*(W_{s^{\text{sshp}}_{K+1-k}})$, then s^{sshp} has no blockage.*

The condition on the sequence s^* stated in Proposition 3.5 is always satsified for $K = 2$ workers. It ensures that for each cycle, each worker starts at their steady-state hand-off position of the order they are currently processing, which guarantees a no-blockage sequence. This holds regardless of the shape of the cumulative workload distribution functions, and for any work rates r.

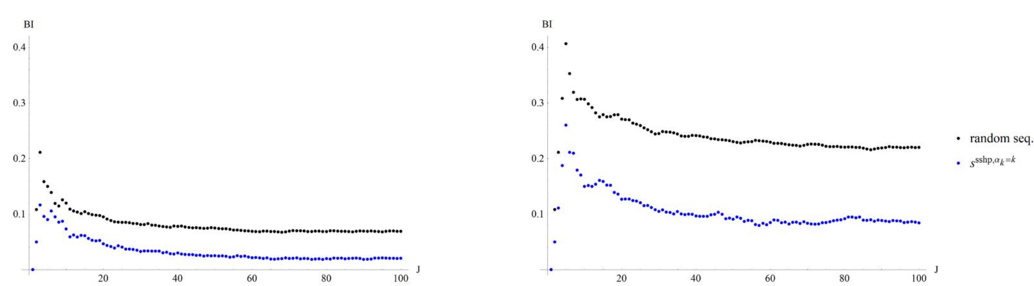
Example 3.2. Consider a BB sequencing problem (W, r), with $r = \left(\frac{9}{16}, 1\right)$ for two workers and maximal work rate ratio $\bar{r} = \frac{9}{16}$, and with four orders to be processed with identical total workload $a = 1$. By Definition 3.1, the respective steady-state hand-off positions are as in Figure 5(a). Then, according to Definition 3.3 and Proposition 3.5, if the orders are sequenced with decreasing $x_1^*(W_j)$, i.e., $s^{\text{sshp}} = (1, 2, 3, 4)$, and the starting position of worker 2 in the first cycle is $x_0(W, r, s^{\text{sshp}}) = x_1^*$, then s^{sshp} has no blockage. The example demonstrates that given the process initialization, orders with more picks at the end of the line (e.g., w_1 in Figure 5(b)) tend to be sequenced early, and *vice versa*.

We demonstrate numerically the sequencing policy s^{sshp}, with $\alpha_k = k$ for $k = 1, \ldots, K-1$, for piecewise linear order types. These order types are motivated by order-picking as described in the Introduction (see Figure 1). Figure 6 depicts the average BI values of 10 randomly sampled sequencing problems with $K=3$, 5 identical workers and increasing number of orders J, where each order has total workload equal to one and is independently drawn with a piecewise linear cumulative workload distribution function (specifically, with approximation levels $L = 6$, $P = 24$; more details about the definition of such order types is provided in Section 3.5). For comparison, a uniformly random sequencing policy generates higher BI values. For $J = 100$, the average BI value with $K = 3$ is 0.069 under a random sequence,



(a) cumulative workload (b) workload density

Figure 5. Four-order example with identical total workload.



(a) $K = 3$ (b) $K = 5$

Figure 6. BI for s^{sshp} and piecewise linear orders with $K=3$, 5 workers.

which reduces to 0.021 under s^{sshp}, i.e., 70% improvement. Again for $J = 100$, the average BI value with $K = 5$ is 0.22 under a random sequence, which reduces to 0.084 under s^{sshp}, i.e. 62% improvement.

3.4. Strong no-blockage sequencing

The no-blockage requirement of Subsection 3.2, despite being useful, is very demanding as it requires no-blockage simultaneously for all possible sequences of a large set of order types. In this subsection we propose and study a weaker requirement based on a simple optimization principle with respect to the starting position of workers $2, \ldots, K$. It is based on the idea that the zero starting position $x_0(W, r, s) = 0$, consequently the first cycle in a sequence, is a blockage worst case.

Definition 3.4. An ordered pair of orders has *strong pairwise no-blockage* given r if no blockage would occur in any cycle of a sequence consisting only of this ordered pair for any pair of consecutive workers working alone. A sequence s has *strong no-blockage* for (W, r) if each consecutive pair in s satisfies strong pairwise no-blockage given r.

A necessary and sufficient condition for strong no-blockage is given by Corollary A.1 in Appendix A. Strong no-blockage is also monotonic with respect to work rate ratios, as shown next.

Proposition 3.6. *Given work rates $r, \hat{r}$ with $\frac{r_k}{r_{k+1}} \leq \frac{\hat{r}_k}{\hat{r}_{k+1}}$ for all $k = 1, \ldots, K-1$ and a set of orders W, if a sequence s has strong no-blockage for $(W, \hat{r})$ then the same holds also for (W, r).*

3.4.1. First-order distributional dominance sequences

Strong no-blockage allows us to propose a sequencing policy based on a dominance relation between order types.

Definition 3.5. For any set of orders W, we say that s is a first-order distributional dominance sequence if $W_{s_{n+1}}(x) \geq W_{s_n}(x)$ for each n and $x \in [0, 1]$.

Since first-order distributional dominance is a relatively weak requirement, the following proposition shows that many sequencing problems are guaranteed to have strong no-blockage.

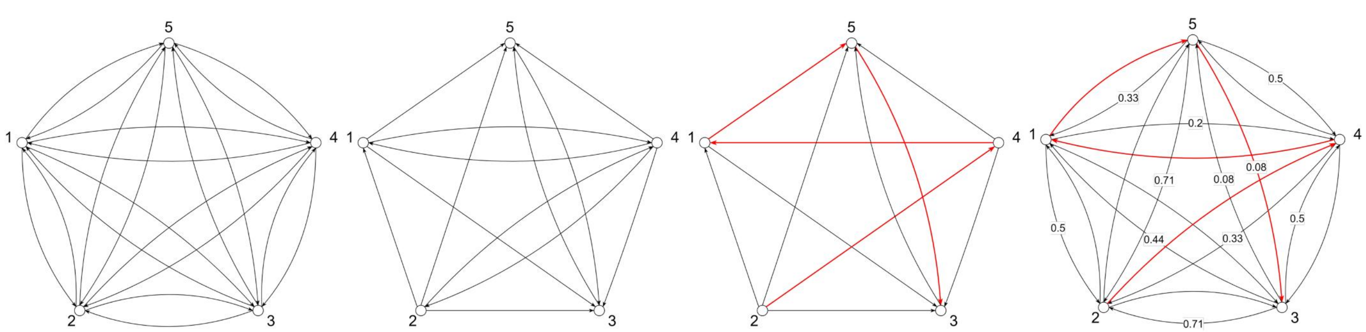
Proposition 3.7. *For any sequencing problem (W, r) with work rate ratio $\bar{r}_k \leq 1$ for all workers $k = 1, \ldots, K-1$, if s is a first-order distributional dominance sequence then s has strong no-blockage for (W, r). When $r_1 = \ldots = r_K$ the converse also holds, i.e., if s has strong no-blockage for (W, r) then s is a first-order distributional dominance sequence.*

3.4.2. Reducing BI using TSP and Hamiltonian paths

We now link the problem of finding a strong no-blockage sequence for (W, r) to the problem of finding a minimum cost TSP and Hamiltonian Paths (see e.g., Garey and Johnson, 1979) in an appropriately defined graph. Given a sequencing problem (W, r), consider a complete directed graph where the set of nodes is the set of orders W, and the cost of the arc connecting order j to order j' is defined as the BI generated by this ordered pair based on strong pairwise no-blockage given r, i.e., the BI of processing only this ordered pair by the consecutive workers $k, k+1$ with highest work rate ratio $\bar{r} \equiv \max_{k=1,\ldots,K-1} \bar{r}_k$ working alone. Then the set of zero cost TSP paths in this graph, or equivalently the Hamiltonian paths in the sub-graph containing only the zero cost arcs, is exactly the set of strong no-blockage sequences for (W, r). It follows by Proposition 3.6 that s is a no-blockage sequence for (W, r) if a corresponding Hamiltonian path exists for some $\hat{r}$ with $\frac{r_k}{r_{k+1}} \leq \frac{\hat{r}_k}{\hat{r}_{k+1}}$ for all $k = 1, \ldots, K-1$. Furthermore, we may define the following policy related to strong no-blockage:

Definition 3.6. Let s^{TSP} be a sequence corresponding to a minimum cost TSP path.

Example 3.3. Figure 7 demonstrates the TSP and Hamiltonian paths approach for the five order (W, r) in Example 3.1. The directed graphs (a)-(c) in the figure are the zero-cost sub-graphs corresponding to the maximal work rate ratios $\bar{r} = 0.15, 0.5, 0.66$, respectively, for $K \geq 2$ workers. As explained in Section 3.2, for $\bar{r} = 0.15$, the set of orders W is a subset of a maximal universal no-blockage set, therefore any sequence has no blockage. Even though this property does not hold for the higher maximal work rate ratios $\bar{r} = 0.5, 0.66$, there are strong no-blockage sequences for (W, r) that correspond to Hamiltonian paths in the graphs shown in Figures 7(a)-(c), for example $s = (2, 4, 1, 5, 3)$. This is the unique such sequence for $\bar{r} = 0.66$,



(a) $\bar{r} = 0.15$ (b) $\bar{r} = 0.5$ (c) $\bar{r} = 0.66$ (d) $\bar{r} = 1$

Figure 7. Five-order strong no-blockage example; directed graphs for $\bar{r} = 0.15, 0.5, 0.66, 1$; no-blockage sequence $(2,4,1,5,3) = s^{\mathrm{TSP}}$.

marked with red arcs in Figure 7(c). By Proposition 3.6, s is a no-blockage sequence for all $\bar{r} \le 0.66$, and calculation shows that this does not hold for any $\bar{r} > \frac{2}{3}$. The graph (d) is the complete directed graph corresponding to the maximal work rate ratio $\bar{r} = 1$, i.e. for identical workers. The sequence $s^{\mathrm{TSP}} = (2,4,1,5,3)$, marked with red arcs also in Figure 7(d), is a minimum cost TSP path in this graph. Calculation using the analysis in Appendix A shows that in fact this is a no-blockage sequence.

3.5. Total workload and steady-state hand-off positions combined sequencing

In this subsection we propose a sequencing policy that generates low BI when the number of orders to be sequenced is relatively large.

Definition 3.7. Let s^{lex} be a sequence in which groups of identical orders are sorted lexicographically according to (i) increasing total workload, and then (ii) decreasing weighted average, $\sum_{k=1}^{K-1} \alpha_k x_k^*$, for increasing weights $0 < \alpha_1 \le \ldots \le \alpha_{K-1}$, of steady-state hand-off positions.

Intuitively, the increasing total workload sequencing policy s^{lex} has the advantage of no-blockage sequencing of consecutive identical orders, as explained in Section 3.1. On top of that, it leads to higher total workload for upstream workers as compared with downstream workers, which tends to reduce the overall physical progress of upstream workers along the line relative to downstream workers. Furthermore, for different orders with identical total workload, s^{lex} leads to processing of orders with decreasing steady-state hand-off positions, with higher weight given to downstream workers. As shown in Section 3.3, such sequencing reduces blockage opportunities. This holds because the difference between the work accomplished by any upstream worker and the work they would accomplish were they to move together with the next downstream worker tends to decrease.

Consider a general, parametric domain of distributions, motivated by order-picking as described in the Introduction (see Figure 1). For a given finite ordered set $\mathcal{X} = \{y_p\}_{p=0}^{P}$ with an integer $P \ge 2$ being the number of pick faces, and where $\mathcal{X}$ is the set of increasing pick face boundary positions, $0 = y_0 < y_1 < y_2 < \ldots < y_P = 1$, let $\mathcal{W}_{\mathcal{X}}$ be the set of all order types $\hat{W}$ such that $\hat{W}$ is piecewise linear with $\mathcal{X}$ being its set of break points, and where $\hat{W}(y_p) \in [0,1]$ for $1 \le p \le P$. When the number of breakpoints P is large, this domain is a good approximation for the domain of all order types normalized to have total workload of at most one. The number of breakpoints P is therefore referred to as a positional approximation level. Each $\hat{W} \in \mathcal{W}_{\mathcal{X}}$ may be represented by the vector $A = [a_1, \ldots, a_P]$ with $0 \le a_1 \le a_2 \le \ldots \le a_P \le 1$, where $\hat{W}(x) = a_p + \frac{a_{p+1} - a_p}{y_{p+1} - y_p}(x - y_p)$ for $x \in [0,1)$ and p such that $y_p \le x < y_{p+1}$, and in particular $a_p = \hat{W}(y_p)$ for all $1 \le p \le P$.

We concentrate on a particular subset of piecewise linear order types, which will be then used to show that s^{lex} is a good sequencing policy. Specifically, for two integers $L \ge 1$ and $Q \ge 2$, where L is a workload approximation level and $P = QL$ is the positional approximation level, consider order types such that each $a_p \in \{0, \frac{1}{L}, \frac{2}{L}, \ldots, 1\}$, and each of these values up to a_P is attained for some p. Figure 8 illustrates such order types with $y_p = \frac{p}{P}$ for all $1 \le p \le P$, and for two approximation specifications alongside an approximated order type with smooth cumulative workload distribution function. The number of layers, L, along the vertical axis in the figure represents the possible positive cumulative workload values (two in the left and six in the right), and the number of breakpoint positions, P, along the horizontal axis represents the number of pick faces. Note that we may partition any such set of order types to groups $l = 0, 1, \ldots, L$, where all orders in group l have identical total workload of $\frac{l}{L}$, and the number of order types in group l is $C_l = \frac{P!}{l!(P-l)!}$, thus the total number of all order types is $C = \sum_{l=0}^{L} C_l$.

The next result shows how sequencing according to s^{lex} reduces the blockage inefficiency for large problems.

Proposition 3.8. *For any sequencing problem (W, r) with piecewise linear W under some workload approximation level L, positional approximation level P and slowest-to-fastest configuration, using the sequencing policy s^{lex}, as the number of orders J approaches $+\infty$, the BI approaches zero.*

Finally for this subsection, we demonstrate the sequencing policy s^{lex} numerically with piecewise linear orders types. Figure 9 depicts the average BI values of 10 randomly sampled sequencing problems with identical workers and increasing number of orders J, where each order is independently drawn with a piecewise linear cumulative workload distribution function given the approximation levels

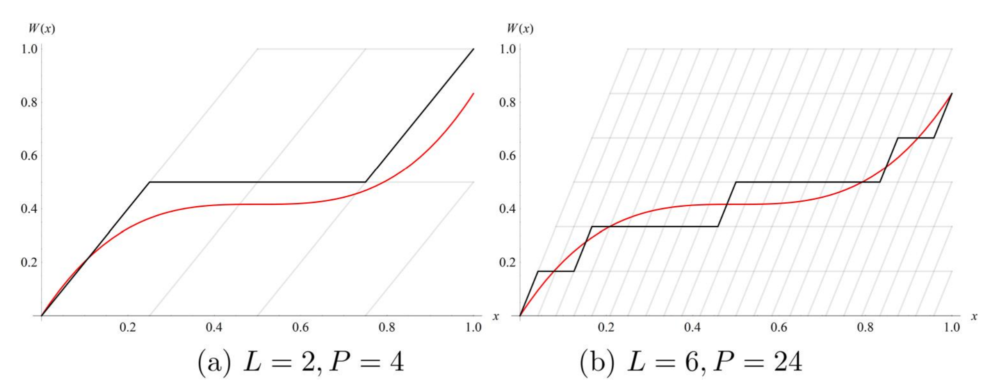


Figure 8. Approximation using piecewise linear distributions.

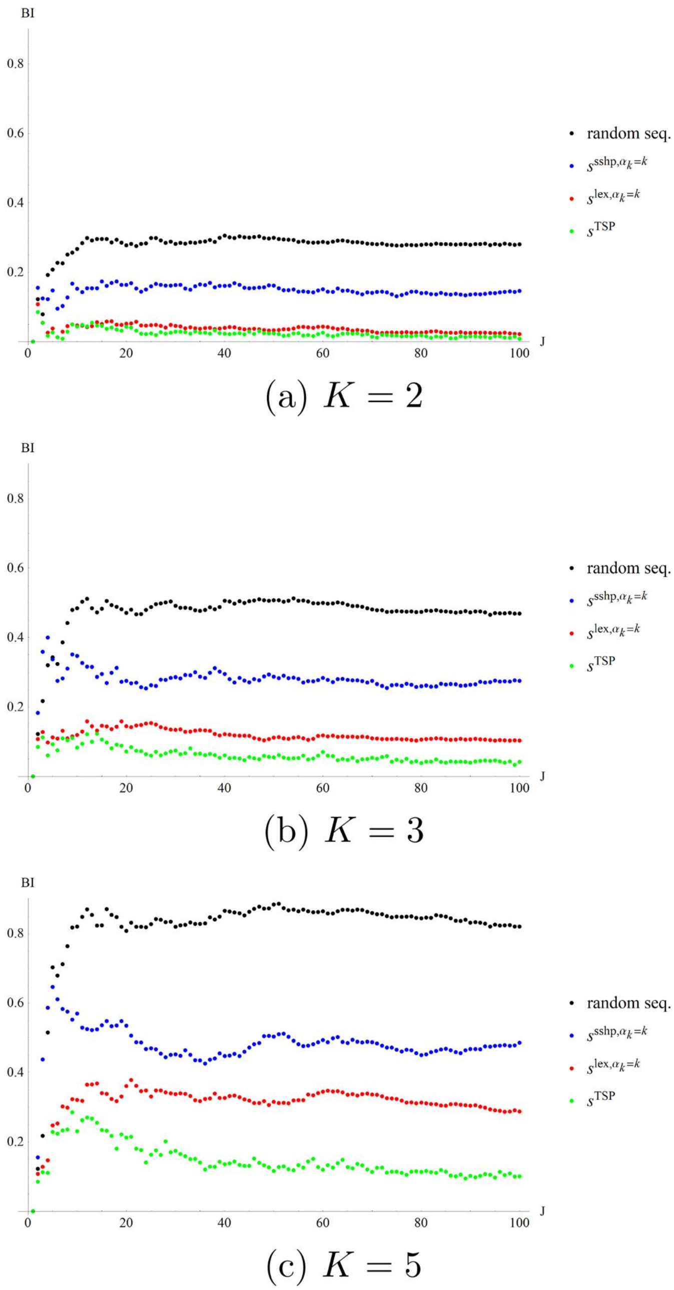


Figure 9. BI for piecewise linear orders with $L=6$ and $P=24$ and $K=2,3,5$ under different sequencing policies.

$L=6$, $P=24$ and $K=2,3,5$ workers, such that the total workload is first uniformly drawn and then the particular type is uniformly drawn among all types with the already given total workload. Using the sequencing policy s^{lex} with $\alpha_k=k$ for $k=1,\ldots,K-1$ leads to decreasing BI values, which are relatively low already for 10 orders. For comparison, a uniformly random sequencing policy generates substantially higher BI values.

Table 1 presents the average BI and MSI values for $J=100$ for each of the sequencing policies and number of workers depicted in Figure 9. We observe that for each sequencing policy, as the blockage opportunities tend to grow, the average BI value increases with the number of workers. Additionally, for each sequencing policy, the percentage improvement in the average BI value compared to random sequencing, as presented in the table, decreases with the number of workers. The improvements increase when proceeding top to bottom in the sequencing policies. The policy $s^{\mathrm{sshp},\alpha_k=k}$ is clearly better than random sequencing. The policy s^{lex} is better than the policy $s^{\mathrm{sshp},\alpha_k=k}$, mainly due to the added lexicographic criterion of the total workload. Note that the latter two policies have polynomial complexity as they are based on sorting principles. The policy s^{TSP} is observed as the best, and should be used whenever the computational burden is worthwhile. As shown in the table, the BI and MSI values agree on the ranking of the policies.

Table 1. BI, MSI and % improvement of each policy compared to random sequencing.

No. of workers		2	3	5
random seq.	BI	0.280	0.469	0.820
	MSI	0.288	0.493	0.881
$s^{\mathrm{sshp},\,\alpha_k=k}$	BI	0.146 (47.9%)	0.276 (41.2%)	0.486 (40.8%)
	MSI	0.149 (48.3%)	0.284 (42.3%)	0.505 (42.8%)
$s^{\mathrm{lex},\,\alpha_k=k}$	BI	0.022 (92.1%)	0.103 (77.9%)	0.287 (65.0%)
	MSI	0.031 (89.1%)	0.122 (75.3%)	0.330 (62.6%)
s^{TSP}	BI	0.008 (97.0%)	0.042 (91.0%)	0.101 (87.7%)
	MSI	0.018 (93.6%)	0.059 (87.9%)	0.138 (84.4%)

4. Concluding remarks

Order heterogeneity in BB order-picking lines may potentially reduce the throughput due to blockages. We provide methods to quantify this inefficiency and propose practical order sequencing policies that substantially reduce it. There are several insights and conclusions. First, identical orders should be picked consecutively to generate zero blockage. Second, when the order workload is controllable, for example via batching, it is recommended to create batches satisfying maximal universal no-blockage for any sequence. Alternatively, one may generate batches of equal total workload and apply the sequencing policy s^{sshp} in order to guarantee no blockage. Yet a further option is to apply the strong no-blockage sequencing policy s^{TSP} when its computational burden is justified. Finally, when batching is not possible or cannot achieve the above conditions, the tractable sequencing policy s^{lex} leads to substantial improvements.

A direction in which our approach may be generalized and extended is to allow for robustness considerations. This may involve accommodating varying work rates across the process cycles. Sequencing in a stochastic environment may be investigated as well. Other future research directions include the investigation of order batching with or without sequencing, and studying the sequencing problem in other order-picking line configurations, such as pick-and-pass (De Koster *et al.*, 2007), cellular BB, and BB with overtaking.

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Appendix A: Preliminary analysis

In this appendix we take the first, technical steps in our analysis, providing a necessary and sufficient condition for no blockage in any cycle, as in Definition 2.4, and for blockage in any cycle. Throughout the analysis, for notational convenience we set $x_{n,0}(W,r,s) = 0$ for any cycle n.

A useful way of understanding our BI measure of inefficiency is through its numerator $BL(W,r,s)$, the work capacity loss due to blockage. First, note that, using Equation (2.1),

$$\begin{aligned} &BL(W,r,s) \\ &= \sum_{n=1}^{J-1} \sum_{k=\max\{1,n+K-J\}}^{K-1} \left(\frac{r_k}{r_K} \left[W_{s_n}(x_{n,K}(W,r,s)) - W_{s_n}(x_{n-1,K-1}(W,r,s)) \right] \right. \\ &\left. - \left[W_{s_{n+K-k}}(x_{n,k}(W,r,s)) - W_{s_{n+K-k}}(x_{n-1,k-1}(W,r,s)) \right] \right), \end{aligned}$$

which, since worker K is never blocked, may be interpreted as the sum, over all busy cycles for workers $k = 1, ..., K-1$, of the difference between the amount of work, $\frac{r_k}{r_K}[W_{s_n}(x_{n,K}(W,r,s)) - W_{s_n}(x_{n-1,K-1}(W,r,s))]$, that can potentially be accomplished by worker k during cycle n, and the amount of work, $W_{s_{n+K-k}}(x_{n,k}(W,r,s)) - W_{s_{n+K-k}}(x_{n-1,k-1}(W,r,s))$, actually accomplished by worker k due to possible blockages during this cycle. In particular, for cycle n with $x_{n-1,k}(W,r,s) = 0$ for any worker $k = 1, ..., K-1$, e.g. $n = 1$, the summand simplifies to

$$\frac{r_k}{r_K}\left[W_{s_n}(x_{n,K}(W,r,s)) - W_{s_n}(x_{n-1,K-1}(W,r,s))\right].$$

Now, for any order j processed by worker $k+1$ immediately followed by order j' processed by worker k and all $x \in [0,1]$, define the cumulative workload difference function

$$d_{j',j,k}(x) \equiv \bar{r}_k W_j(x) - W_{j'}(x).$$

To interpret this difference, consider some cycle n with $x_{n-1,k}(W,r,s) = 0$ for any worker $k = 1, ..., K-1$, where worker $k+1$ holds order $j = s_{n-1+K-k}$ and worker k holds order $j' = s_{n+K-k}$. Suppose first that no blockage occurs for any position of worker $k+1$ within the interval $I = (0,x)$ for some $x \in (0,1]$. Then $d_{j',j,k}(x)$ is the difference between the work accomplished by worker k throughout interval I of worker $k+1$, and the work worker k would accomplish were worker k to reach position x together with worker $k+1$. In this case, $d_{j',j,k}(x) \leq 0$, and interval I has zero contribution to the work capacity loss due to blockage.

Formally, to study no-blockage sequences, the following lemma is key.

Lemma A.1. *Given a sequencing problem (W,r), a sequence s has no-blockage if and only if for all cycles $n = 1, 2, ..., J-1$, worker $k \in \{\max\{1, n+K-J\}, ..., K-1\}$ and $x \in [x_{n-1,k}(W,r,s), x_{n,k+1}(W,r,s)]$,*

$$d_{s_{n+K-k}, s_{n-1+K-k}, k}(x) \leq \bar{r}_k\left[W_{s_{n-1+K-k}}(x_{n-1,k}(W,r,s)) - W_{s_{n+K-k}}(x_{n-1,k-1}(W,r,s))\right]. \tag{A1}$$

Note that Lemma A.1 implies that if no blockage occurs throughout cycle n when the starting position of the workers is $x_{n-1}(W,r,s)$, then no blockage would continue to hold throughout the cycle if the starting position of any single worker $k = 2, ..., K$ was larger. Consequently, the zero starting position of all workers is the worst case in terms of blockage. This implies that the first cycle in a sequence, for which we always assume $x_{0,k}(W,r,s) = 0$ for all $k = 1, ..., K-1$, is in particular a blockage worst case.

In cycle n of a no-blockage sequence s, the position of hand-off n satisfies $x_{n,K}(W,r,s) = 1$ and the recursive relation

$$\begin{aligned} x_{n,k}(W,r,s) = \max\{x | W_{s_{n+K-k}}(x) - W_{s_{n+K-k}}(x_{n-1,k-1}(W,r,s)) \\ = \bar{r}_k(W_{s_{n-1+K-k}}(x_{n,k+1}(W,r,s)) - W_{s_{n-1+K-k}}(x_{n-1,k}(W,r,s)))\} \end{aligned} \tag{A2}$$

for $k = 1, ..., K-1$. In this recursion, the max is relevant when the equality constraint in (A2) holds over a closed interval of x values, which happens when $W_{s_{n+K-k}}(x)$ is constant there because the workload density is zero.

The following corollary to Lemma A.1 characterizes strong no-blockage (see Definition 3.4).

Corollary A.1. *A sequence s has strong no-blockage for (W,r) if and only if for all cycles $n = 1, 2, ..., J-1$ and all workers $k \in \{\max\{1, n+K-J\}, ..., K-1\}$,*

$$\max_{0 \leq x \leq 1} d_{s_{n+K-k}, s_{n-1+K-k}, k}(x) \leq 0.$$

Suppose now that worker k is the most downstream worker blocked by worker $k+1$, where this blockage occurs for the entire position interval $I = (0,x)$ for some $x \in (0,1]$. Then $d_{j',j,k}(x)$ is the difference between the amount of work, $\bar{r}_k W_j(x)$, that can potentially be accomplished by worker k within interval I, and the amount of work, $W_{j'}(x)$, actually accomplished by worker 1 due to this blockage. In this case, $d_{j',j}(x) > 0$, and is exactly the positive contribution of interval I to the work capacity loss due to blockage.

To understand the possibility of blockages in general, we may consider intervals $I = (y_1, y_2) \subseteq [x_{n-1,k}(W,r,s), x_{n,k+1}(W,r,s)]$ for $x_{n-1,k}(W,r,s) \geq 0$, where y_1 is a joint position for workers $k, k+1$ if one exists otherwise $y_1 = x_{n-1,k}(W,r,s)$, and let $y_0 = y_1$ in the former case and $y_0 = x_{n-1,k-1}(W,r,s)$ in the latter. Then, extending the function $d_{j',j,k}(x)$ to such intervals I by defining $d_{j',j,k}(I) = d_{j',j,k}(y_2) - d_{j',j,k}(y_1)$, no blockage occurs for any position of worker $k+1$ within the interval I if and only if $d_{j',j,k}(I') \leq \bar{r}_k[W_j(y_1) - W_j(y_0)]$ for all intervals $I' = (y_0, x)$ with $x \leq y_2$, in which case interval I has zero contribution to the work capacity loss due to blockage. Additionally, blockage occurs throughout I if and only if $y_0 = y_1$ and $d_{j',j,k}(I') > 0$ for all subintervals $I' \subseteq I$, in which case $d_{j',j,k}(I)$ is exactly the contribution of the interval I to the work capacity loss due to blockage. In the latter case, $d_{j',j,k}(I)$ is the difference between the amount of work, $\bar{r}_k(W_j(y_2) - W_j(y_1))$, that can be potentially accomplished by worker 1 during interval I, and the amount of work, $W_{j'}(y_2) - W_{j'}(y_1)$, actually accomplished by worker 1 due to this blockage. It follows that $BL(W,r,s)$, the work capacity loss due to blockage and the numerator of $BI(W,r,s)$, is equal to the sum of $d_{j',j,k}(I)$ over all disjoint blockage intervals in all cycles.

Formally, to study blockage sequences, the following lemma is key.

Lemma A.2. *Given a sequencing problem (W,r), a sequence s and a cycle $n = 1, 2, ..., J-1$, when worker $k \in \{\max\{1, n+K-J\}, ..., K-1\}$ is the most downstream worker blocked by worker $k+1$, this blockage occurs at position $x \in (x_{n-1,k}(W,r,s), x_{n,k+1}(W,r,s))$ if and only if*

$$\begin{aligned} d_{s_{n+K-k}, s_{n-1+K-k}, k}(x) - d_{s_{n+K-k}, s_{n-1+K-k}, k}(l_k(x)) > \bar{r}_k\left[W_{s_{n-1+K-k}}(l_{k+1}(x))\right. \\ \left. - W_{s_{n-1+K-k}}(l_k(x))\right] \end{aligned} \tag{A3}$$

and

$$d'_{s_{n+K-k}, s_{n-1+K-k}, k}(x^-) > 0, \tag{A4}$$

where $d'_{s_{n+K-k}, s_{n-1+K-k}, k}(x^-)$ is the left derivative of $d_{s_{n+K-k}, s_{n-1+K-k}, k}(x)$ at x, and either $l_k(x) = l_{k+1}(x)$ is the joint position of the two workers during cycle n at the end of the previous blockage interval, or, in the case where there is no such previous blockage interval, $l_k(x) = x_{n-1,k-1}(W,r,s)$ and $l_{k+1}(x) = x_{n-1,k}(W,r,s)$ are the initial positions of worker k and worker $k+1$, respectively, at the beginning of cycle n.

Example A.1. Consider a BB sequencing problem (W,r), with identical work rates that are equal to one for two workers, i.e., $r = (1,1)$ and $\bar{r}_1 = \frac{r_1}{r_2} = 1$, and with two piecewise linear orders (see Section 3.5) to be processed, $W = (W_j)_{j=1,2}$, with joint break points $\mathcal{X} = \{\frac{1}{3}, \frac{2}{3}, 1\}$ and representing vectors $A_1 = [\frac{1}{3}, \frac{1}{2}, 1]$ and $A_2 = [\frac{1}{6}, \frac{1}{2}, \frac{2}{3}]$. Consider the sequence $s = \{1,2\}$, thus $j = 1$ and $j' = 2$. The following Table 2 presents the functions used in Lemma A.2 in order to characterize the blockage obtained. Since the first cycle ends with blockage, the entire work is completed in a single cycle. As shown in the table, several intervals are distinguished, depending on whether Conditions (A3) or (A4) are met, which in turn determine two blockage intervals during the single cycle, one interval at the beginning and the other at the end

Table 2. The functions used in Lemma A.2.

interval	$x \in (0, \frac{1}{3})$	$x \in [\frac{1}{3}, \frac{2}{3})$	$x \in [\frac{2}{3}, \frac{5}{6}]$	$x \in (\frac{5}{6}, 1]$
$W_j(x)$	x	$\frac{x}{2} + \frac{1}{6}$	$\frac{3x}{2} - \frac{1}{2}$	$\frac{3x}{2} - \frac{1}{2}$
$W_{j'}(x)$	$\frac{x}{2}$	$x - \frac{1}{6}$	$\frac{x}{2} + \frac{1}{6}$	$\frac{x}{2} + \frac{1}{6}$
$d_{j',j,1}(x)$	$\frac{x}{2}$	$-\frac{x}{2} + \frac{1}{3}$	$x - \frac{2}{3}$	$x - \frac{2}{3}$
$l_1(x) = l_2(x)$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Cond. (A3)	True	False	False	True
Cond. (A4)	True	False	True	True
Blockage	True	False	False	True
Contribution to BI	$\frac{1}{10}$	0	0	$\frac{1}{10}$

of the cycle. The contribution of each blockage interval I to $BI(W,r,s)$ is $\frac{d_{1,2}(I)}{TW(W)} = \frac{1/6}{1+2/3} = \frac{1}{10}$.

Lemma A.2 may also be used to analyze general blockage cases, in which worker k is not the most downstream worker blocked by worker $k+1$. This is done by defining for each blocked worker $k+1$ a virtual order type with workload distribution $\hat{W}$ increased from their actually processed order type W exactly so that this worker is no longer blocked by the next downstream worker $k+2$. Then Lemma A.2 may be applied recursively from worker $K-1$ back to worker 1 using virtual order types to test whether each worker k, now potentially the most downstream blocked worker, is actually blocked by worker $k+1$.

The following result provides an exact analytic expression, in the case of quadratic distributions and two workers, for the work capacity loss due to blockage, thus for the BI, and for the hand-off positions for each cycle n.

Proposition A.1. *For a sequencing problem (W,r) with two workers, quadratic W_j and a sequence s, the work capacity loss due to blockage $BL(W,r,s) = \sum_{n=1}^{J-1} \Delta_n$, where Δ_n is recursively given, together with the workload ω_n accomplished by worker 1 during cycle $n = 1,2,\ldots,J-1$, by* the displayed Δ_n, ω_n

$$(\Delta_n, \omega_n) = \begin{cases} (d^{\max}_{j',j,1} - \bar{r}_1\omega_{n-1}, \bar{r}_1 W_j(1) - d^{\max}_{j',j,1}), & 0 < d'_{j',j,1}(0) < -d''_{j',j,1}(0), d^{\max}_{j',j,1} > \bar{r}_1\omega_{n-1} \\ (d_{j',j,1}(1) - \bar{r}_1\omega_{n-1}, W_{j'}(1)), & \neg(0 < d'_{j',j,1}(0) < -d''_{j',j,1}(0), d^{\max}_{j',j,1} > \bar{r}_1\omega_{n-1}) \\ & \quad \text{and } d_{j',j,1}(1) > \bar{r}_1\omega_{n-1} \\ (0, \bar{r}_1(W_j(1) - \omega_{n-1})), & \text{otherwise,} \end{cases}$$

for $j = s_n$, $j' = s_{n+1}$, $d^{\max}_{j',j,1} \equiv d_{j',j,1}\left(\frac{d'_{j',j,1}(0)}{-d''_{j',j,1}(0)}\right)$ and $\omega_0 = 0$.

Additionally, the hand-off positions $x_n(W,r,s)$ satisfy $\omega_n = W_{j'}[x_n(W,r,s)]$, and whenever $\Delta_n > 0$, the unique blockage interval $I_n = (y_{n1}, y_{n2})$ during cycle n satisfies $\Delta_n = d_{j',j,1}(I_n) = d_{j',j,1}(y_{n2}) - d_{j',j,1}(y_{n1})$ for y_{n1} satisfying $d_{j',j,1}(y_{n1}) = \bar{r}_1\omega_{n-1}$ and

$$y_{n2} = \begin{cases} \dfrac{d'_{j',j,1}(0)}{-d''_{j',j,1}(0)}, & 0 < d'_{j',j,1}(0) < -d''_{j',j,1}(0), d^{\max}_{j',j,1} > \bar{r}_1\omega_{n-1} \\ 1, & \text{otherwise.} \end{cases}$$

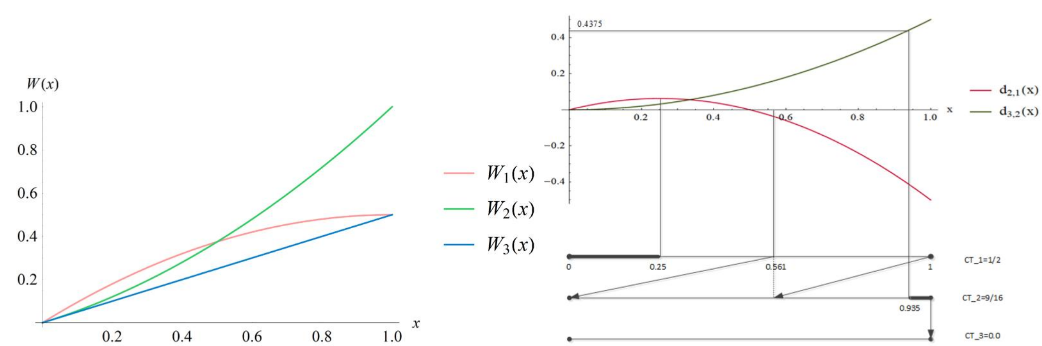
The following example demonstrates Proposition A.1 and summarizes the preliminary analysis of this section.

Example A.2. Consider a BB sequencing problem (W,r), with identical work rates that are equal to one for two workers, i.e., $r = (1,1)$ and $\bar{r}_1 = r_1/r_2 = 1$, and with three orders to be processed, $W = (W_j)_{j=1,2,3}$, such that

$$W_1(x) = x\left(1 - \frac{x}{2}\right), W_2(x) = x\left(\frac{1+x}{2}\right), W_3(x) = \frac{x}{2},$$

as shown in the left graph of Figure 10.

Suppose that the chosen processing sequence for these orders is $1 \to 2 \to 3$, i.e., $s = (1,2,3)$. Then, in the first cycle, worker 2 processes order 1 and worker 1 processes order 2, and both start at position $x = 0$, i.e., $x_0(W,r,s) = 0$. Cycle 1 has cycle time of $CT_1 = \frac{W_1(1) - W_1(0)}{r_2} = \frac{1}{2}$. For this cycle, the cumulative workload difference function $d_{2,1,1}(x) = \bar{r}_1 W_1(x) - W_2(x) = x\left(\frac{1}{2} - x\right)$, as shown in Figure 10, is concave and achieves its maximal value of $d^{\max}_{2,1,1} = \frac{1}{16} > 0$ at $x^{\max} = \frac{1}{4}$. Therefore, worker 1 has their first blockage interval $I_1 = (0, x^{\max})$ starting immediately at the beginning of the cycle and ending at position $x^{\max}$, with no blockage occurring from that point until the end of the cycle, and the contribution of this blockage interval to the work capacity loss $BL(W,r,s)$ is $\Delta_1 = d_{2,1,1}(I_1) = d^{\max}_{2,1,1} - d_{2,1,1}(0) = \frac{1}{16}$ (the details for this calculation are given in Proposition A.1). Worker 1 ends the cycle at the position x for which their cumulative workload distribution function $W_2(x)$ is equal to $\omega_1 \equiv \bar{r}_1 W_1(1) - d^{\max}_{2,1,1} = \frac{7}{16}$, i.e. $x = \frac{3\sqrt{2}-2}{4} \approx 0.561$. So this is the hand-off position $x_{1,1}(W,r,s)$ at which worker 2 takes order 2 from worker 1, worker 1 takes order 3 at position $x = 0$, and cycle 2 initiates. It follows that cycle 2 has cycle time of $CT_2 = \frac{W_2(1) - W_2(x_1(W,r,s))}{r_2} = \frac{W_2(1) - \omega_1}{r_2} = \frac{9}{16}$. For this cycle, the cumulative workload difference function $d_{3,2,1}(x) = \bar{r}_1 W_2(x) - W_3(x) = \frac{x^2}{2}$ is convex with $d_{3,2,1}(1) > \bar{r}_1\omega_1 = \frac{7}{16}$ and $d_{3,2,1}\left(\sqrt{\frac{7}{8}}\right) = \frac{7}{16}$, therefore worker 1 has their second blockage interval $I_2 = \left(\sqrt{\frac{7}{8}}, 1\right)$ starting from position $x = \sqrt{\frac{7}{8}} \approx 0.935$ and ending at position $x = 1$ at the end of the cycle, thus the contribution of this blockage interval to the work capacity loss $BL(W,r,s)$ is $\Delta_2 = d_{3,2,1}(I_2) = d_{3,2,1}(1) - d_{3,2,1}\left(\sqrt{\frac{7}{8}}\right) = \frac{1}{16}$ (again, this follows from Proposition A.1). Consequently, the second hand-off



(a) cumulative workload (b) cummulative workload difference

Figure 10. Three-order blockage example.

position is $x_{2,1}(W,r,s) = 1$, at which point both workers complete processing their respective orders. Cycle 3 therefore has cycle time $CT_3 = \frac{W_3(1) - W_3(1)}{1} = 0$.

To summarize the example, the MS for this sequence is $\sum_{n=1}^{3} CT_n = 1\frac{1}{16}$, the total workload of the three orders is $TW(W) = \sum_{j=1}^{3} W_j(1) = 2$, and the makespan work capacity is $MC(W,r,s) = \left(\frac{1}{2} + \frac{9}{16}\right)(1+1) + 0 \cdot 1 = 2\frac{1}{8}$, therefore the work capacity loss due to blockage is $BL(W,r,s) = MC(W,r,s) - TW(W) = \frac{1}{8}$. Note that the two blockage intervals I_1, I_2 contribute equally to $BL(W,r,s)$. Finally, the blockage inefficiency for this sequence is $BI = \frac{1}{16}$.

Appendix B: Proofs

Proof of Proposition 2.1. Using Definitions 2.2 and 2.3, for any sequencing problem (W,r) and any sequence s,

$$BI(W,r,s) = \frac{MC(W,r,s)}{TW(W)} - 1 = \frac{\sum_{n=1}^{J} CT_n(W,r,s)\left(\sum_{k=\max\{1,\,n+K-J\}}^{K} r_k\right)}{TW(W)} - 1 \le \frac{\sum_{n=1}^{J} CT_n(W,r,s) \sum_{k=1}^{K} r_k}{TW(W)} - 1 \tag{B1}$$

$$= \frac{MS(W,r,s)}{TW(W)/\sum_{k=1}^{K} r_k} - 1 = MSI(W,r,s).$$

Denote by s^{BI} the sequence that minimizes the BI and by s^{MS} the sequence that minimizes the MS (and the MSI). Therefore,

$$\begin{aligned}
0 &\le MSI(W,r,s^{\mathrm{MS}}) - BI(W,r,s^{\mathrm{MS}}) \\
&\le MSI(W,r,s^{\mathrm{MS}}) - BI(W,r,s^{\mathrm{BI}}) \\
&= \min_s MSI(W,r,s) - \min_s BI(W,r,s) \\
&\le \left(\frac{\sum_{n=1}^{J} CT_n(W,r,s^{\mathrm{BI}})}{TW(W)/\sum_{k=1}^{K} r_k} - 1\right) - \left(\frac{\sum_{n=1}^{J} CT_n(W,r,s^{\mathrm{BI}}) \frac{\sum_{k=\max\{1,\,n+K-J\}}^{K} r_k}{\sum_{k=1}^{K} r_k}}{TW(W)/\sum_{k=1}^{K} r_k} - 1\right) \\
&= \frac{\sum_{n=\max\{1,J-K+2\}}^{J} CT_n(W,r,s^{\mathrm{BI}})\left(1 - \frac{\sum_{k=n+K-J}^{K} r_k}{\sum_{k=1}^{K} r_k}\right)}{TW(W)/\sum_{k=1}^{K} r_k} \\
&= \frac{\sum_{n=\max\{1,J-K+2\}}^{J} \frac{W_{s_n^{\mathrm{BI}}}(1) - W_{s_n^{\mathrm{BI}}}(x_{n-1,K-1}(W,r,s^{\mathrm{BI}}))}{r_K}\left(1 - \frac{\sum_{k=n+K-J}^{K} r_k}{\sum_{k=1}^{K} r_k}\right)}{TW(W)/\sum_{k=1}^{K} r_k} \\
&\le \frac{\sum_{n=\max\{1,J-K+2\}}^{J} \frac{W_{s_n^{\mathrm{BI}}}(1)}{r_K}}{TW(W)/\sum_{k=1}^{K} r_k} = \left(\sum_{k=1}^{K} \frac{r_k}{r_K}\right) \sum_{n=\max\{1,J-K+2\}}^{J} \frac{W_{s_n^{\mathrm{BI}}}(1)}{TW(W)} \\
&\le \left(\sum_{k=1}^{K} \frac{r_k}{r_K}\right) \min\{J, K-1\} \frac{\max_{1\le j\le J} W_j(1)}{TW(W)},
\end{aligned}$$

where the first inequality is inequality (B1) for $s = s^{\mathrm{MS}}$, and the second and third inequalities follow from the definitions of $s^{\mathrm{MS}}, s^{\mathrm{BI}}$ and Definition 2.3 since $BI(W,r,s^{\mathrm{BI}}) \le BI(W,r,s^{\mathrm{MS}})$ and $MS(W,r,s^{\mathrm{MS}}) \le MS(W,r,s^{\mathrm{BI}})$, respectively. The two subsequent equalities follow since the first $\max\{0, J-K+1\}$ terms of each of the two sums over cycles n in the fourth line in this chain of inequalities are identical, and from Equation (2.1), respectively. The fourth inequality follows by omitting the two negative terms, and the last inequality follows by replacing each total workload by the maximal total workload over all orders in the sequence. Finally, when fixing a positive lower and upper bound on each order's total workload, the last expression in this chain approaches zero as the number of orders increases. Therefore, the difference $\min_s MSI(W,r,s) - \min_s BI(W,r,s)$ approaches zero since it is non-negative and has an upper bound that approaches zero. It follows that when the number of orders is large, any sequence that minimizes the BI over all sequences approximately achieves the minimum MSI over all sequences. Moreover, since the total workload $TW(W)$ and the work rates r are independent of the sequence of orders, any such sequence also approximately achieves the minimum MS over all sequences. ∎

Proof of Proposition 2.2. By Equation (2.1), since $CT_n(W,r,s) \le \frac{W_{s_n}(1)}{r_K}$ for any cycle n, Equation (2.3) implies

$$MC(W,r,s) \le \sum_{n=1}^{J} \frac{W_{s_n}(1)}{r_K} \sum_{k=1}^{K} r_k = \frac{\sum_{k=1}^{K} r_k}{r_K} TW(W),$$

thus $BI(W,r,s) \le \frac{\sum_{k=1}^{K-1} r_k}{r_K}$. To see that this bound is tight, consider the K-order sequencing problem where $W_1(x)$ has $W_1(1) > 0$ and $W_j(x) = 0$ for $j = 2,\ldots,J$ and $x \in [0,1]$ and the chosen processing sequence is $1 \to 2 \to \ldots \to K$, i.e. $s = (1,2,\ldots,J)$. Then, in the first cycle worker K processes order 1 and determines $CT_1(W,r,s) = \frac{W_1(1) - 0}{r_K} = \frac{W_1(1)}{r_K}$, workers $1,\ldots,K-1$ process orders $2,\ldots,J$, respectively, and, being blocked throughout the cycle, end at $x_1(W,r,s) = 1$. All the remaining cycles $n = 2,\ldots,J$ have $CT_n(W,r,s) = \frac{0-0}{r_K} = 0$. Therefore,

$$BI(W,r,s) = \frac{\frac{W_1(1)}{r_K} \cdot \sum_{k=1}^{K} r_k + 0}{W_1(1) + 0} - 1 = \frac{\sum_{k=1}^{K-1} r_k}{r_K}.$$

Proof of Lemma A.1. Consider any cycle $n = 1,2,\ldots,J-1$ and worker $k \in \{\max\{1, n+K-J\},\ldots,K-1\}$, and let $j = s_{n-1+K-k}$ and $j' = s_{n+K-k}$. Condition (A1) is equivalent to $\bar{r}_k[W_j(x) - W_j(x_{n-1,k}(W,r,s))] \le W_{j'}(x) - W_{j'}(x_{n-1,k-1}(W,r,s))$, which means that if workers $k, k+1$ were to reach position $x \in [x_{n-1,k}(W,r,s), x_{n,k+1}(W,r,s)]$ from their respective starting positions, namely $x_{n-1,k-1}(W,r,s))$ for worker k and $x_{n-1,k}(W,r,s)$ for worker $k+1$, the amount of work, $\bar{r}_k[W_j(x) - W_j(x_{n-1,k}(W,r,s))]$, that could be potentially accomplished by worker k is at most the required amount of work, $W_{j'}(x) - W_{j'}(x_{n-1,k-1}(W,r,s))$, i.e., worker k is not blocked by worker $k+1$ up to position x. Since this holds for all $x \in [x_{n-1,k}(W,r,s), x_{n,k+1}(W,r,s)]$, no blockage occurs throughout the cycle. ∎

Proof of Lemma A.2. Let $j = s_{n-1+K-k}$ and $j' = s_{n+K-k}$. Condition (A3) is equivalent to $\bar{r}_k(W_j(x) - W_j(l_2(x))) > W_{j'}(x) - W_{j'}(l_1(x))$, which means that while the workers are moving to position $x \in (x_{n-1,k}(W,r,s), x_{n,k+1}(W,r,s))$ from their respective starting positions, $l_k(x)$ for worker k, the amount of work, $\bar{r}_k(W_j(x) - W_j(l_2(x)))$, that can be potentially accomplished by worker k is larger than the required amount of work, $W_{j'}(x) - W_{j'}(l_1(x))$. This is necessary for blockage to occur at position x, as otherwise worker k will not be able to reach the position of worker $k+1$. Adding Condition (A4), both conditions are together sufficient for blockage at position x because Condition (A4) means that this positive work difference is increasing also at position x. Note that even if Condition (A3) holds, violation of Condition (A4) implies that blockage does not occur at x. This is the case because Condition (A3) implies that the two workers are positioned at x simultaneously, but violation of Condition (A4) implies that such blockage will no longer hold when the two workers infinitesimally proceed with their work. This shows that the two conditions are together necessary and sufficient for blockage to hold at position x. ∎

Proof of Proposition 3.1. Consider any sequence s. Since for each cycle $n = 1,2,\ldots,J-1$, worker $k \in \{\max\{1, n+K-J\},\ldots,K-1$ and $j = s_{n-1+K-k}$ and $j' = s_{n+K-k}$,

$$\begin{aligned}
\max_{0\le x\le 1} d_{j',j,k}(x) &= \max_{0\le x\le 1} \left(\bar{r}_k \bar{W}_j(x) - \bar{W}_{j'}(x)\right) \\
&\le \max_{0\le x\le 1} \left(\bar{W}(x) - \bar{W}(x)\right) = 0 \\
&\le \bar{r}_k \bar{W}(x_{n-1,k}(\bar{W},r,s) - \bar{W}(x_{n-1,k-1}(\bar{W},r,s))),
\end{aligned}$$

the conclusion follows from Lemma A.1. ∎

Proof of Proposition 3.2. Define the function $f : [0, \bar{W}(1)]^{K+1} \to [0, \bar{W}(1)]^{K+1}$ by $f_0(y) = 0$, $f_K(y) = \bar{W}(1)$ and $f_k(y) = y_{k-1} + \frac{r_k}{r_K}[\bar{W}(1) - y_{K-1}]$ for all $k = 1, \ldots, K-1$ and all y. By Equation (A2), the function f maps the cumulative workloads at hand-off $n-1$ to the cumulative workloads at hand-off n, i.e., $(\bar{W}[x_{n,k}(\bar{W}, r, s)], k = 0, \ldots, K) = f(\bar{W}[x_{n-1,k}(\bar{W}, r, s)], k = 0, \ldots, K)$. Note that for $\bar{r}_k < 1$ for all $k = 1, \ldots, K-1$, $f(y)$ is a contraction mapping because

$$\begin{aligned}|f(y^1) - f(y^2)| &= \sum_{k=1}^{K-1}\left|(y^1_{k-1} - y^2_{k-1}) - \frac{r_k}{r_K}(y^1_{K-1} - y^2_{K-1})\right| \\ &\le \sum_{k=1}^{K-2}\left|(y^1_k - y^2_k) + \left(\frac{\max_{k'=1,\ldots,K-1} r_{k'}}{r_K}\right)|y^1_{K-1} - y^2_{K-1}|\right| \\ &\le \sum_{k=1}^{K-2}|y^1_k - y^2_k| + \left(\frac{\max_{k=1,\ldots,K-1} r_k}{r_K}\right)|y^1_{K-1} - y^2_{K-1}| \\ &< \sum_{k=1}^{K-1}|y^1_k - y^2_k| \\ &= |y^1 - y^2|.\end{aligned}$$

Applying a fixed-point theorem, the hand-off position $x_n(\bar{W}, r, s)$ converges to the steady-state hand-off position $x^*(\bar{W})$ as the number of orders J approaches infinity, and the convergence rate is exponential. ■

Proof of Proposition 3.3. Fix a sequence s. To see (a), note that for any pair of orders processed in cycle n with order types $W_j, W_{j'} \in \mathcal{H}_{\bar{r}, W^0}$ for $j = s_{n-1+K-k}$ processed by worker $k \in \{\max\{1, n+K-J\}, \ldots, K-1\}$ and $j' = s_{n+1}$ by worker $k+1$,

$$\begin{aligned}&\max_{x_{n-1,k}(W,r,s) \le x \le x_{n,k}(W,r,s)} d_{j',j,k}(x) \\ &= \max_{x_{n-1,k}(W,r,s) \le x \le x_{n,k}(W,r,s)} \left(\bar{r}_k W_j(x) - W_{j'}(x)\right) \\ &\le \max_{x_{n-1,k}(W,r,s) \le x \le x_{n,k}(W,r,s)} \left(\bar{r} W^0(x) - \bar{r} W^0(x)\right) \\ &= 0 \le \bar{r}_k\left[W_j(x_{n-1,k}(W,r,s)) - W_{s_{n+K-k}}(x_{n-1,k-1}(W,r,s))\right],\end{aligned}$$

which implies by Lemma A.1 that no blockage occurs during cycle n.

To see (b), first note that for any order type $\hat{W} \notin \mathcal{H}_{\bar{r}, W^0}$, there exists $0 < x' \le 1$ such that either (i) $\hat{W}(x') > W^0(x')$, or (ii) $\hat{W}(x') < \bar{r} W^0(x')$. In case (i), considering $W = (\hat{W}, \bar{r}_{K-1} W^0)$ with $j = s_1 = 1$ for worker $k = K-1$ and $j' = s_2 = 2$ for worker K,

$$\begin{aligned}&\max_{0 \le x \le 1} d_{j',j,k}(x) \ge d_{j',j,k}(x') = \bar{r}_k \hat{W}(x') - \bar{r}_k W^0(x') > \bar{r}_k W^0(x') \\ &-\bar{r}_k W^0(x') = 0,\end{aligned}$$

thus, since $x_{0,k}(W, r, s) = 0$ for all $k = 0, \ldots, K-1$, by Lemma A.1, there is blockage during the first cycle. In case (ii), denoting by k the most downstream worker for which $\bar{r}_k = \bar{r}$, and considering $W = (W^0, \ldots(K-k \text{ times})\ldots, W^0, \hat{W})$ with $j = s_{1+K-k} = 1+K-k$ for worker k and $j' = s_{2+K-k} = 2+K-k$ for worker $k+1$,

$$\begin{aligned}&\max_{0 \le x \le 1} d_{j',j,k}(x) \ge d_{j',j,k}(x') = \bar{r}_k W^0(x') - \hat{W}(x') > \bar{r} W^0(x') \\ &-\bar{r} W^0(x') = 0,\end{aligned}$$

thus again, by Lemma A.1, there is blockage during the first cycle. ■

Proof of Proposition 3.4. The direction that if the algorithm concludes the answer yes then it is in fact yes follows directly from Proposition 3.3. For the other direction, suppose that the algorithm concludes the answer no, i.e., there exists an order $W_{j'}$ in the set W and position $x' \in [0, 1]$ such that $W_{j'}(x') < \bar{r} W^0(x')$. Let j be an order in W such that $W_j(x') = W^0(x')$, denote by k the most downstream worker for which $\bar{r}_k = \bar{r}$, and consider a sequence s in which the first $K-k+1$ orders are $(W_j, \ldots(K-k \text{ times})\ldots, W_j, W_{j'})$. Then

$$\begin{aligned}&\max_{0 \le x \le 1} d_{j',j,k}(x) \ge d_{j',j,k}(x') = \bar{r} W_j(x') - W_{j'}(x') > \bar{r} W^0(x') \\ &-\bar{r} W^0(x') = 0,\end{aligned}$$

thus, by Lemma A.1, there is blockage during the first cycle. By Definition 3.2, W cannot in fact be a subset of some maximal universal no-blockage set because this would imply that any sequence s is a no-blockage sequence for (W, r). This proves the other direction. ■

Proof of Proposition 3.5. By definition, a sequence s^{sshp} is sorted decreasingly by $\sum_{k=1}^{K-1} \alpha_k x^*_k$ for some $\alpha_{K-1} \ge \ldots \ge \alpha_1 > 0$. Since $x^*_k(W_{s^*_1}) \ge \ldots \ge x^*_k(W_{s^*_J})$ for all workers $k = 1, \ldots, K$, $x^*_k(W_{s^{\mathrm{sshp}}_1}) \ge \ldots \ge x^*_k(W_{s^{\mathrm{sshp}}_J})$ for all workers $k = 1, \ldots, K$. In fact, s^* is a s^{sshp} sequence for all $\alpha_{K-1} \ge \ldots \ge \alpha_1 > 0$. Note that all cycles $n = 1, 2, \ldots, J-1$ have the same cycle time of

$$CT_n = \frac{a - W_{s^{\mathrm{sshp}}_n}(x^*_{K-1}(W_{s^{\mathrm{sshp}}_n}))}{r_K} = \frac{a - a\frac{\sum_{k'=1}^{K-1} r_{k'}}{\sum_{k=1}^{K} r_k}}{r_K} = \frac{a}{\sum_{k=1}^{K} r_k}.$$

At any such cycle n, each worker $k = 2, \ldots, K$ starts at the $(k-1)$th component of the steady-state hand-off position of order $s^{\mathrm{sshp}}_{K+n-k}$, $x^*_{k-1}(W_{s^{\mathrm{sshp}}_{K+n-k}})$ for which

$$W_{x^*_{K+n-k}}(x^*_{k-1}(W_{s^{\mathrm{sshp}}_{K+n-k}})) = a\frac{\sum_{k'=1}^{k-1} r_{k'}}{\sum_{k=1}^{K} r_k},$$

processes this order, and ends at the kth component of the steady-state hand-off position of this order, $x^*_k(W_{s^{\mathrm{sshp}}_{K+n-k}})$ for which

$$W_{x^*_{K+n-k}}(x^*_k(W_{s^{\mathrm{sshp}}_{K+n-k}})) = a\frac{\sum_{k'=1}^{k} r_{k'}}{\sum_{k=1}^{K} r_k}.$$

The fact that the orders are with decreasing $x^*_k(W_j)$ for all workers $k = 1, \ldots, K-1$ indeed ensures that no blockage occurs in any cycle. ■

Proof of Proposition 3.6. This follows by Corollary A.1 since for every cycle $n = 1, \ldots, J-1$, all workers $k \in \{\max\{1, n+K-J\}, \ldots, K-1\}$, and all x and $j = s_{n-1+K-k}$ and $j' = s_{n+K-k}$, $d_{j',j,k}(x)$ is increasing in the work rate ratio $\bar{r}_k$, so if the ordered pair W_j followed by $W_{j'}$ has strong pairwise no-blockage given some $\bar{r}_k$ then this will hold also for any smaller $\bar{r}_k$. ■

Proof of Proposition 3.7. Suppose that s is a first-order distributional dominance sequence. Since for each cycle $n = 1, 2, \ldots, J-1$, all workers $k \in \{\max\{1, n+K-J\}, \ldots, K-1\}$, and $j = s_{n-1+K-k}$ and $j' = s_{n+K-k}$,

$$\begin{aligned}\max_{0 \le x \le 1} d_{j',j,k}(x) &= \max_{0 \le x \le 1}\left(\bar{r}_k W_j(x) - W_{j'}(x)\right) \\ &\le \max_{0 \le x \le 1}\left(W_j(x) - W_{j'}(x)\right) \le 0,\end{aligned}$$

the conclusion follows from Corollary A.1. That the converse holds when $r_1 = \ldots = r_K$ follows from Definition 3.5. ■

Proof of Proposition 3.8. Fix L, P and a sequencing problem (W, r) with piecewise linear W under L, P. Let C be the total number of order types given L, P, and recall that each order type has total workload of at most one. Any sequence s^{lex} consists of at most C subsequences of consecutive identical orders. By Proposition 3.7, since $\bar{r}_k \le 1$ for $k = 1, \ldots, K-1$, the number of blockage cycles under s^{lex} is at most C. Each such cycle n contributes to $BL(W, \hat{r}, s^{\mathrm{lex}})$ at most

$$CT_n(W, r, s^{\mathrm{lex}})\sum_{k=1}^{K-1} r_k \le \frac{W_{s^{\mathrm{lex}}_n}(1)}{r_K}\sum_{k=1}^{K-1} r_k \le \sum_{k=1}^{K-1}\frac{r_k}{r_K} \le K-1.$$

Therefore $BI(W, r, s^{\mathrm{lex}}) \le \frac{C(K-1)}{TW(W)}$, of which the right-hand side approaches zero as J approaches $+\infty$ since its numerator does not depend on J and its denominator approaches $+\infty$. ■

Proof of Proposition A.1. .

Consider $K = 2$ workers and cycle n, and denote $j = s_n$ and $j' = s_{n+1}$. By Lemma A.2, blockage occurs during cycle n if and only if there exists $x \in (x_{n-1}(W, r, s), 1)$ such that $d_{j'j}(x) > \bar{r}_1 W_j(x_{n-1}(W, r, s))$ and $d'_{j'j}(x) > 0$. For quadratic orders, the cumulative workload difference function $d_{j'j}(x)$ is the quadratic function

$$x\left(\bar{r}_1 b_j - b_{j'} + \left(\bar{r}_1 (a_j - b_j) - (a_{j'} - b_{j'})\right)x\right).$$

Therefore blockage occurs during cycle n if and only if one of the following two cases holds: (i) $[0 < d'_{j',j}(0) < -d''_{j',j}(0)$ and $d^{\max}_{j',j} > \bar{r}_1 W_j(x_{n-1}(W,r,s))]$, or (ii) [not (i) and $d_{j'j}(1) > \bar{r}_1 W_j(x_{n-1}(W,r,s))]$. In both cases there is a unique blockage interval $[\bar{x}, \tilde{x}]$ satisfying $d_{j',j}(\bar{x}) = \bar{r}_1 W_j(x_{n-1}(W,r,s))$, and $\tilde{x} = \frac{d'_{j'j}(0)}{-d''_{j'j}(0)} < 1$ in case (i) and $\tilde{x} = 1$ in case (ii). Therefore the hand-off position of cycle n is $x_n(W,r,s) = 1$ in case (ii), and is determined in case (i) from the no-blockage interval $[\tilde{x}, 1]$ such that $W_{j'}(x_n(W,r,s)) - W_{j'}(\tilde{x}) = \bar{r}_1(W_j(1) - W_j(\tilde{x}))$. The contribution of cycle n to $BL(W,r,s)$ is $\Delta_n \equiv d_{j',j}(\tilde{x}) - d_{j',j}(\bar{x})$, thus substitution of $\bar{x}, \tilde{x}$ for the various cases establishes the corresponding expressions as given in the proposition. A similar substitution also establishes the expressions for ω_n, the workload accomplished by worker 1 during cycle n. ■