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# Decentralization Cost in Scheduling: A Game-Theoretic Approach

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Decentralized organizations may incur inefficiencies because of scheduling issues associated with competition among decision makers (DMs) for limited resources. We analyze the *decentralization cost* (DC), i.e., the ratio between the Nash equilibrium cost and the cost attained at the centralized optimum. Solution properties of a dispatching-sequencing model are derived and subsequently used to develop bounds on the DC for an arbitrary number of jobs and DMs. A scheduling-based coordinating mechanism is then provided, ensuring that the centralized solution is obtained at equilibrium.

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## 1. Introduction

Decision-making processes in industrial and service organizations are often decentralized. The cost involved in collecting, processing, and analyzing large amounts of data for all required decisions make centralized control very difficult in many cases. As a result, many layers of decision makers (DMs) can generally be found in a medium or large organization, handling tactical production processes, all the way up to the strategic boardroom-type issues. Frequently, DMs operate and are measured according to some incentive scheme. DMs aim at achieving their own objectives; consequently, conflicts are sometimes inevitable in a decentralized environment.

One of the causes of conflict between DMs is the existence of competition for common resources with limited capacity. For example, these resources could be financial or time resources of both key employees or central work centers/machines. In such cases, DMs try to use the resources for their own benefit, and the result does not necessarily correspond to the maximum benefit of the organization. It may happen that DMs cooperate for their mutual benefit, but this is not necessarily always true.

The competition for resources is often associated with scheduling problems. For example, suppose that jobs scheduled on a common resource/machine are associated with different DMs, each having an interest in optimizing the performance measure of their work (e.g., minimum flow time). In service organizations, each customer aims at minimizing her own waiting time and in production systems, jobs (or sets of jobs) belonging to different people in the organization may compete for time on a common machine.

Common resources usually operate under a centralized sequencing rule. This rule is determined either to maintain a fair environment (e.g., first in first out (FIFO)) or to optimize the system's overall performance, usually by minimizing some cost-oriented performance measure (e.g., shortest processing time (SPT) to minimize flow time). If the two aims are in conflict, one may suggest that the optimal rule ought to be implemented, while the gain obtained by applying this rule (SPT) instead of the fair rule (FIFO) should be fairly divided among the jobs/people based on their final versus initial situation in the sequence. The literature suggests that cooperative games could model such situations. Curiel et al. (1989) initiated this line of research by proposing the one-machine sequencing game, in which several jobs, each belonging to a different agent, have to be processed on a single machine. Each job is defined by a processing time, and the cost of each job is assumed to be linear in completion time. The initial sequence is not necessarily optimal regarding the total cost. The authors suggest that an allocation rule induced by a cooperative game could allocate the cost saving that results from coalitions' decisions to move from the initial sequence to an optimal one. Other sequencing games can be found in the literature, covering both the single-machine case with different assumptions to those of Curiel et al. (1989) and the multimachine case.

On a more general level than sequencing, various scheduling models have been proposed in the literature in the context of cooperative games. Tijs et al. (1984) proposed a permutation game model, a generalization of the assignment game of Shapley and Shubik (1971), assigning jobs to machines with several players, each having one machine and one job. Each machine is able to process a single job, and each job can be processed by any of the machines, although not necessarily requiring the same duration. Thus, assignment of jobs to machines may involve gains that can be distributed between the players. Interested readers may refer to cooperative scheduling models surveyed in Borm et al. (2001) and Curiel et al. (2002).

Despite the relatively substantial research concerning cooperative games and scheduling, the literature on noncooperative games in a similar environment is scarce, although it could be argued that DMs may not cooperate with others when making decisions. This stands in contrast to the large body of literature on noncooperative models of decentralized, coordinated supply chains (see, e.g., Cachon 2003). In one notable paper, Hain and Mitra (2004) analyze a sequencing model with a set of agents, each having one job to process on a common facility, where the job duration is privately known only to the agent. The central manager cannot monitor the agents' actions or verify the true processing durations without costly information gathering, thus facing the agents' incentive to announce low durations to advance in the queue under an SPT rule. To solve this problem, the paper suggests a generalized Vickrey-Clarke-Groves (VCG) mechanism, which is based on money transfers that depend on the agents' announcements as to job durations. Under general conditions, they show it is possible to design these money transfers to be noncostly for the manager (by always adding up to zero) and to give the agents an incentive to reveal their true job durations, thus enabling the centralized optimum.

The literature has shown that a centralized optimum inducing mechanism is not always possible. Jéhiel and Moldovanu (2001) prove, in general settings that may involve scheduling situations as a special case, that when agents' private information is multidimensional, e.g., when an agent may have more than one job to process, no mechanism can implement a centralized optimum. Moreover, irrespective of single or multidimensional private information, even in cases where such mechanisms do exist, it is possible that none of them allow money transfers that add up to zero (i.e., are noncostly to the central manager).

In this paper, we refer to a situation that involves noncooperative games and scheduling. Each DM associated with a different department in a firm aims at optimizing some performance measure, while taking into account the possible strategies of other DMs. The total cost associated with the game is expected to be higher than the optimal cost associated with a centralized optimum. The ratio between the two, entitled *decentralization cost* (DC), is the main subject of analysis, and contribution, of this paper.[1] It is important for a central manager to know the DC, as this would make clear the extent to which it is worthwhile investing effort to approach a centralized optimum. We present a general model for decentralized scheduling systems, in which a coordinating mechanism can be implemented when the centralized solution is not a Nash equilibrium. We present a dispatching-sequencing game and characterize the properties of the decentralized and centralized solutions for an arbitrary number of jobs and DMs and use these properties to analyze the DC. We also provide a scheduling-based coordinating mechanism for this model, which ensures that the centralized solution is obtained at equilibrium. Such a mechanism should be employed after considering its implementation cost versus the DC. Contrary to the existing literature in scheduling and supply chain, we introduce mixed Nash equilibria into the analysis.

In the next section, we discuss decentralized modelling of scheduling environments in general. In §3, a two-alternative resources model is presented with some illustrative examples, while §4 analyses the

[1] Lariviere and Porteus (2001) and Cachon (2003) refer to a similar ratio of profits as an efficiency measure of a decentralized supply chain.

properties of the centralized and decentralized solutions. Bounds for the DC are presented in §5. A coordinating mechanism is given in §6 and a summary in §7.

## 2. Competition over Resources in Noncooperative Decentralized Scheduling

Traditional scheduling problems assume a centralized system associated with a single DM. A general definition of scheduling problems is given in Pinedo (2002, p. 1), who states that "Scheduling deals with the allocation of scarce resources to tasks over time. It is a decision-making process with the goal of optimizing of one or more objectives." In practice, several DMs may compete for the capacity of a single resource or set of resources, whilst optimizing their own performance measures/objectives, sometimes to the detriment of the firm. In a decentralized system, the derived solution may impair the overall performance of the system, associated with its global objectives. Consequently, the general director in charge of the DMs must decide whether to enforce her authority to achieve an optimal decision, or alternatively to provide a mechanism that changes the incentive schemes of the individuals to encourage the separate parts to achieve as close to the optimal decision as possible. This decision ought to be dependent on the DC.

To compute the DC, one should analyze both the centralized and decentralized systems. The former is associated with an optimization problem with the objective of minimizing the total cost of the firm. For the analysis of the decentralized system, we propose applying noncooperative game theory tools, specifically the Nash equilibrium solution (Nash 1951, see also Osborne and Rubinstein 1994). In the decentralized problem, the set of decision variables is partitioned to $n+1$ subsets that represent the DMs/players. It is assumed that each DM $1, \dots, n$ has full authority to make decisions independently of the others. We also assume that the decisions made by DMs $1, \dots, n$ do not necessarily exhaust all the decisions made by the centralized system. The remaining decisions are comprised within DM 0, which represents lower-level DMs that take as given the decisions made by DMs $1, \dots, n$. For example, the decisions made by DMs $1, \dots, n$ may be which jobs to send to the competing resource(s), while the remaining decisions made by DM 0 may relate to the sequencing of these jobs. Thus, each player is interested in improving private benefits and the resulting outcome is associated with a Nash equilibrium. Note that DM 0's decision is made given the other DMs' decisions. Thus, each DM $1, \dots, n$ foresees the decision made by DM 0 as a result of her own decision.

It is possible that no pure Nash equilibrium exists, in which case a mixed Nash equilibrium will exist (Nash 1951). In a mixed Nash equilibrium, each DM chooses a mixed strategy, i.e., chooses a probability distribution over pure strategies. Each DM's objective is then to minimize the expected cost, calculated over all possible combinations that the DMs may choose and assuming independence between the probabilities they assign.

Clearly, a decentralized equilibrium is a feasible solution of the centralized problem. Thus, the cost of a decentralized equilibrium is at least as high as that of a centralized optimal solution, consequently, the ratio of two such values, the DC, is a unitless measure having value of at least one. If the centralized solution is achieved at equilibrium, we say that the system is coordinated. In this case, even when there are multiple equilibria in the system, one can assume that the centralized solution is attainable, because only a single effort is required to bring the system to the centralized equilibrium. Once a DM understands that the others act according to the centralized solution, the DM will not deviate from it. Consequently, because all realize this, the centralized solution will indeed occur. Hence, we distinguish between a situation in which the centralized solution is an equilibrium and one where it is not. In the latter case, a mechanism, as explained above, should be considered based on the DC versus the cost of implementing the mechanism. When the DC cannot be explicitly calculated, bounds can be used to estimate the extent to which the decentralized solution is more costly than the centralized solution. Such bounds and mechanism are discussed below after introducing the DC in our model.

## 3. A Model of Two Alternative Resources

In this section, we model a dispatching-sequencing problem, where each of several departments has a set of jobs. Jobs can be processed on a common resource with limited capacity, entitled hereafter the *in-house* machine, or on an alternative resource, entitled the *subcontractor*. The latter has an unlimited capacity and guarantees any job $j$ with completion time of $kt_j$ (in parallel), where $t_j$ is job $j$'s in-house processing time and $k$ is a factor greater than or equal to one. The decision to be made by each department manager (DM) regarding each job is whether to utilize the in-house common resource, without controlling the other departments, or to use the alternative resource and send the job to the subcontractor.

The objective function of each department, as well as of the firm, is to minimize the total flow time (we assume that the ready time of all jobs is equal to zero, so *flow time* and *completion time* are used interchangeably). To this end, an SPT rule is applied for sequencing the jobs on the in-house machine. Note that it is also possible to interpret $k$ as including additional job duration proportional costs charged by the subcontractor, under the assumption that the in-house costs are also proportional to jobs' completion times. For this interpretation, the objective function would be minimizing cost rather than flow time.

The assumption that the subcontractor has unlimited capacity is adequate when the subcontractor's available capacity, related to the type of jobs in question, is significantly higher than the firm's required capacity. In this case, a contract can be signed that clearly defines a due-date commitment on the subcontractor's side, which is expressed by the value of $k$. Such due-date performance can be guaranteed because jobs are processed in parallel by the subcontractor. Examples for such agreements can be found in different areas in manufacturing. For example, in flash memory manufacturing, because of the rapidly changing technology, it is crucial to minimize flow time to achieve a good time-to-market performance. Various business units of a given firm may produce different products, e.g., cell phones, MP3 players, digital cameras, etc. It is common that these units compete on joint resources such as assembly and test. When these resources are heavily occupied, the business units can send the jobs to another, larger producer that will process these jobs according to a prior due-date contract. To retain the contract, the subcontractor may apply to other subcontractors in cases of high-demand peaks.

Formally, suppose that a firm must process a set $C = \{1, \ldots, c\}$ of jobs. Each job $j \in C$ can be processed sequentially with duration $t_j > 0$ on a common in-house resource. We assume that durations are distinct and that jobs are indexed in an ascending order of durations, i.e., $j > j' \Rightarrow t_j > t_{j'}$. The firm is divided into departments, $N = \{1, \ldots, n\}$, and each job, $j$, belongs to a single department $i \in N$, i.e., $C = \bigcup_{i \in N} C_i$, where $C_i \cap C_{i'} = \varnothing$ for $i \neq i'$. A subcontractor offers to process any job $j$ with completion time $kt_j$, for a common factor $k \geq 1$. Each department must decide on (mixed) strategies $\mathscr{M}_i = (M_{ih_i}, p_{ih_i})_{h_i}$, i.e., with probability $p_{ih_i}$, set $M_{ih_i} \subseteq C_i$ of jobs is to be processed in-house, where the complementary set $C_i \backslash M_{ih_i}$ of jobs are sent to the subcontractor. A pure strategy, i.e., a strategy for which $p_{ih_i} = 1$ for some $h_i$, is written for short as $M_i$. A profile of strategies for all departments other than $i$, $(\mathscr{M}_{i'})_{i' \neq i}$, is denoted by $\mathscr{M}_{-i}$ and is denoted by $M_{-i}$ if these strategies are pure. Given any subset of jobs $C' \subseteq C$, denote its cardinality by $|C'|$, and given any $1 \leq m \leq |C'|$, denote by $t_{C'}(m)$ the duration of the $m$th shortest job in $C'$. Each department $i$ is interested in the sum of completion times for all jobs in $C_i$. Following an SPT processing strategy for the jobs processed in-house, this sum is calculated by

$$\begin{aligned} F_i(M_i, M_{-i}) = &\sum_{m=1}^{|M_i|} (|M_i| + 1 - m) t_{M_i}(m) \\ &+ \sum_{j \in \cup M_{-i}} |\{j' \in M_i \mid j' > j\}| t_j + k \sum_{j \in C_i \backslash M_i} t_j. \end{aligned} \tag{1}$$

The first term accounts for the total flow time of $i$'s in-house jobs given an SPT rule, while ignoring the other departments' in-house jobs. The second term takes into account the additional flow time of $i$'s in-house jobs due to the other departments' in-house jobs. Namely, the duration of each job $j$ in $M_{-i}$ is added to each of the jobs $j'$ in $M_i$ that are scheduled after job $j$. The third term computes flow times of $i$'s jobs that are sent to the subcontractor. Given $M_{-i}$, department $i$'s objective in a pure Nash equilibrium is $\min_{M_i} F_i(M_i, M_{-i})$. Similarly, for mixed strategies,

$$\begin{aligned} F_i(\mathscr{M}_i, \mathscr{M}_{-i}) &= \sum_{(h_{i'})_{i'}} \left( \prod_{i'} p_{i' h_{i'}} \right) F_i(M_{ih_i}, (M_{i' h_{i'}})_{i' \neq i}) \\ &= \sum_{h_1} \cdots \sum_{h_n} (p_{1h_1} \cdots p_{nh_n}) F_i(M_{1h_1} \cdots M_{nh_n}). \end{aligned} \tag{2}$$

**Table 1** Pure Nash Equilibria for $k = 1.5$

| $M_1 \backslash M_2$ | $\varnothing$ | $\{2\}$ |
|---|---|---|
| $\varnothing$ | (9, 3) | (9, 2) |
| $\{1\}$ | (8.5, 3) | **(8.5, 3)** |
| $\{3\}$ | (6.5, 3) | **(8.5, 2)** |
| $\{1, 3\}$ | (7, 3) | (9, 3) |

Therefore, given $\mathscr{M}_{-i}$, department $i$'s objective in a mixed Nash equilibrium is $\min_{\mathscr{M}_i} F_i(\mathscr{M}_i, \mathscr{M}_{-i})$.

The firm is interested in the sum of completion times for all departments,

$$\begin{aligned} F_0((M_i)_i) &\equiv \sum_i F_i(M_i, M_{-i}) \\ &= \sum_{m=1}^{|\cup_i M_i|} \left( \left| \bigcup_i M_i \right| + 1 - m \right) t_{\cup_i M_i}(m) \\ &\quad + k \sum_{j \in C \backslash \cup_i M_i} t_j. \end{aligned} \tag{3}$$

Therefore, the firm's objective is $\min_{(M_i)_i} F_0((M_i)_i)$, or the corresponding expectation for mixed strategies. Note that the SPT processing strategy mentioned above is a result of the firm's DM 0 objective (see previous section). In other words, the lower-level DMs who take as given the decisions made by DMs $1, \ldots, n$ always follow SPT to minimize the sum of completion times for all departments.

A decentralized decision is represented by a Nash equilibrium $(\mathscr{M}_i^D)_i = (M_{ih_i}^D, p_{ih_i}^D)_{i, h_i}$, whereas a centralized decision is represented by a firm's optimal strategy combination $(\mathscr{M}_i^C)_i = (M_{ih_i}^C, p_{ih_i}^C)_{i, h_i}$. Given a Nash equilibrium and an optimal strategy combination, the corresponding DC is defined as the ratio between the firm's objective value in the former with respect to the latter.

Definition 1. Given a decentralized decision $(\mathscr{M}_i^D)_i$ and an optimal decision $(\mathscr{M}_i^C)_i$, $\mathrm{DC} = F_0((\mathscr{M}_i^D)_i)/F_0((\mathscr{M}_i^C)_i)$.

Example 1. Suppose that there are three jobs ($c = 3$), two departments ($n = 2$), and in-house job durations $t_j$ are 1, 2, 5. Jobs 1 and 3 belong to department 1 and job 2 belongs to department 2, such that $C_1 = \{1, 3\}$ and $C_2 = \{2\}$. Suppose further that the subcontractor offers $k = 1.5$, i.e., processing with completion times 50% higher than in-house job durations. An enumeration of all possible pure-strategy combinations is given in Table 1, where each entry depicts the pair $(F_1(M_1, M_2), F_2(M_1, M_2))$.

**Table 2** Mixed Nash Equilibrium for $k = 1.4$

| $M_1 \backslash M_2$ | $\varnothing$ | $\{2\}$ |
|---|---|---|
| $\varnothing$ | (8.4, 2.8) | (8.4, 2) |
| $\{1\}$ | **(8, 2.8)** | **(8, 3)** |
| $\{3\}$ | **(6.4, 2.8)** | **(8.4, 2)** |
| $\{1, 3\}$ | (7, 2.8) | (9, 3) |

For example, if department 1 processes longer job 3 in-house and sends the shorter job 1 to the subcontractor, i.e., $M_1 = \{3\}$; and department 2 performs job 2 in-house, i.e., $M_2 = \{2\}$, then according to Equation (1), the sum of completion times are $F_1(M_1, M_2) = (1 \cdot 5) + (1 \cdot 2) + (1.5 \cdot 1) = 8.5$, $F_2(M_1, M_2) = (1 \cdot 2) = 2$ and $F_0(M_1, M_2) = 10.5$. According to Table 1, $M_1 = \varnothing$, $M_1 = \{1, 3\}$ are strictly dominated strategies, because whatever department 2 does, department 1 prefers $M_1 = \{3\}$, i.e., to process only its longest job in-house and send the shorter job to the subcontractor. Similarly, $M_2 = \varnothing$ is weakly dominated, because processing job 2 in-house may only improve department 2's objective value, depending on department 1's strategy. Also, note that there is a unique optimal centralized strategy combination $M_1^C = \{3\}$ and $M_2^C = \varnothing$, so only the longest job is processed in-house and the other shorter jobs are sent to the subcontractor. However, this strategy combination is not a Nash equilibrium, because department 2 strictly prefers to process job 2 in-house, thus reducing its objective value to $F_2(\{3\}, \{2\}) = 2$. If department 2 chooses the latter strategy, the resulting combination $M_1^D = \{3\}$ and $M_2^D = \{2\}$ is a Nash equilibrium because department 1 does not strictly prefer to change its strategy. This leads to a DC of $10.5/9.5 \simeq 1.105$. However, there is another Nash equilibrium, $M_1^D = \{1\}$ and $M_2^D = \{2\}$, with a higher DC $11.5/9.5 \simeq 1.211$. The sum of completion times of both pure Nash equilibria are marked in Table 1 with bold letters. Note that a small perturbation of $k$ to a higher value will leave only the former equilibrium, with no change in the centralized solution and only a small perturbation in the value of the firm's objective and the DC. However, for any value of $1 < k < 1.5$, there exists no pure Nash equilibrium. For example, Table 2 corresponds to $k = 1.4$.

In this case, the strictly dominated strategies do not change for department 1; however, there are no dominated strategies for department 2. The centralized solution does not change and is still not an equilibrium for the same reason as before. However, none of the previous equilibria remain; $M_1 = \{3\}$, $M_2 = \{2\}$ is not an equilibrium because department 1 strictly prefers to perform only the shorter job in-house in order to be first in the queue; $M_1 = \{1\}$, $M_2 = \{2\}$ is not an equilibrium because department 2 strictly prefers to send job 2 to the subcontractor because it is no longer the first in the queue. Finally, $M_1 = \{1\}$, $M_2 = \varnothing$ is not an equilibrium because department 1 strictly prefers to change and process only the longer job in-house. For this example, a mixed equilibrium should be expected. There is a unique mixed equilibrium, in which department 1 performs only job 1 in-house with probability 0.8 and performs only job 3 in-house with probability 0.2. Simultaneously, department 2 performs job 2 in-house with probability 0.8 and sends it to the subcontractor with probability 0.2. The sum of completion times of the strategy combinations chosen with positive probability are marked in Table 2 with bold letters. Then department 1 expects a total completion time of 8 when $M_1 = \{1\}$ and expects $0.2 \cdot 6.4 + 0.8 \cdot 8.4 = 8$ when $M_1 = \{3\}$, with only at least as high total completion times when choosing any other mixed strategy, so there is no strict preference to change the probabilities or the actions. Similarly, department 2 expects a total completion time of 2.8 when $M_2 = \varnothing$ and expects $0.8 \cdot 3 + 0.2 \cdot 2 = 2.8$ when $M_2 = \{2\}$, so again there is no strict preference to change the mixed strategy. In this mixed equilibrium, the firm's expected decentralized objective value is $0.8 \cdot 0.2 \cdot (8+2.8) + 0.2 \cdot 0.2 \cdot (6.4+2.8) + 0.8 \cdot 0.8 \cdot (8+3) + 0.2 \cdot 0.8 \cdot (8.4+2) = 10.8$, and the centralized objective value is 9.2, so the DC is $10.8/9.2 \simeq 1.174$.

In the case where $k = 1$ or $k \geq c$, the best equilibrium strategy is straightforward. In the former case, all jobs are sent to the subcontractor, while in the latter, all jobs are processed in-house. Consequently, from now on, we assume that $1 < k < c$.

## 4. Solution Properties

In this section, we analyze the properties of the solution to the problem formulated in the previous section. Apart from providing insights to the strategies chosen by the departments, these properties are useful for later analysis of the DC. First, the properties of a decentralized solution are analyzed, i.e., a Nash equilibrium. Using these properties, a complete characterization of the centralized solution is then provided.

### 4.1. Decentralized Solution Properties

The following proposition states two properties of a Nash equilibrium, in terms of bounds for the number of jobs processed in-house. These bounds depend on the relative efficiency of the two alternative resources, represented by the parameter $k$. First, for any (mixed) Nash equilibrium, each department performs at most $\lfloor k \rfloor$ jobs in-house.[2] This property is a consequence of dominance considerations—i.e., irrespective of the opponents' strategies—because any strategy in which more than $\lfloor k \rfloor$ jobs of $i$ are processed in-house is strictly dominated by a strategy that differs only by sending the shortest in-house job to the subcontractor. This is true because the shortest in-house job contributes its processing duration times the number of $i$'s jobs in-house, which is at least $\lfloor k \rfloor + 1$. This value is greater than $k$, the job duration multiplier when the job is processed by the subcontractor. These dominance considerations allow the first part of the proposition to hold for any best-response strategy, with no dependence on the other players' strategies. The second part of the proposition is a property that depends on the other players' strategies, and therefore is guaranteed for pure Nash equilibria. It states that the total number of jobs that are processed in-house by all departments is bounded from below by $\lfloor k \rfloor$. This holds because if the total number of jobs in-house is lower then $\lfloor k \rfloor$, then at least one player prefers to switch any job from the subcontractor to be processed in-house.

Proposition 1. (1) *Let $\mathcal{M}_i = (M_{ih_i}, p_{ih_i})_{h_i}$ be a mixed strategy for department $i$ such that $|M_{ih_i}| > \lfloor k \rfloor$ for some $h_i$. Then $\mathcal{M}_i$ is a strictly dominated strategy.*

(2) *Let $(M_i^*)_i$ be any pure Nash equilibrium. If $k = \lfloor k \rfloor$, then $|\bigcup_i M_i^*| \geq \lfloor k \rfloor - 1$ and if $k > \lfloor k \rfloor$, then $|\bigcup_i M_i^*| \geq \lfloor k \rfloor$. Moreover, if $k = \lfloor k \rfloor$, there exists a pure Nash equilibrium $(M_i^{**})_i$ such that $|\bigcup_i M_i^{**}| \geq \lfloor k \rfloor$.*

The proof is in the appendix. For the next proposition, we need to define a segment as a sequence of

[2] For any real number $y$, $\lfloor y \rfloor$ denotes the largest integer smaller than or equal to $y$.

jobs that are adjacent and belong to a single player. Segments are derived from durations and ownership of jobs after they are sorted according to their durations. Let each $C_i$, player $i$'s set of jobs, be partitioned to $l_i$ segments, denoted by $S_i^s$ for $s \in \{1, \ldots, l_i\}$. More explicitly, let $C_i = \bigcup_{1 \le s \le l_i} S_i^s$, where $S_i^s \ne \varnothing$ and $S_i^s \cap S_i^{s'} = \varnothing$ for $s \ne s'$ such that the following three conditions hold for each segment $s$: (1) if $j \in C_i$ and $\min\{j' \in S_i^s\} \le j \le \max\{j' \in S_i^s\}$, then $j \in S_i^s$, (2) $\min\{j' \in S_i^s\} - 1 \notin C_i$, and (3) $\max\{j' \in S_i^s\} + 1 \notin C_i$. The following proposition states that by dominance considerations, i.e., irrespective of the opponents' strategies, each department prefers processing only the longest jobs in each segment in-house, while sending the remaining jobs in the segment to the subcontractor. In particular, it is possible that all jobs in some segment are sent to the subcontractor. The proof is in the appendix.

Proposition 2. *Let $\mathcal{M}_i = (M_{ih_i}, p_{ih_i})_{h_i}$ be a mixed strategy for department $i$, satisfying $j \notin M_{ih_i}$ and $j' \in M_{ih_i}$ for some $h_i$ and some jobs $j > j'$, where $j, j' \in S_i^s$ for some $s \in \{1, \ldots, l_i\}$. Then $\mathcal{M}_i$ is a dominated strategy.*

### 4.2. Centralized Solution Properties

The properties of the centralized solution ($n = 1$) can now be established as a corollary to Propositions 1 and 2. In particular, it is optimal for the firm to keep the $\lfloor k \rfloor$ longest jobs in-house and send the rest to the subcontractor. When all departments act accordingly, their strategy profile is entitled the *centralized solution* and each department is said to choose a *centralized strategy*. The proof is in the appendix.

Corollary 1. *Suppose that $n = 1$. Then*

(1) *For any optimal strategy $M_1$, $|M_1| \le \lfloor k \rfloor$.*

(2) *For any optimal strategy $M_1$, if $k = \lfloor k \rfloor$, then $|M_1| \ge \lfloor k \rfloor - 1$; and if $k > \lfloor k \rfloor$; then $|M_1| \ge \lfloor k \rfloor$.*

(3) *There exists an optimal strategy $M_1^*$ such that $|M_1^*| = \lfloor k \rfloor$.*

(4) *The strategy $M_1^* \equiv \{c + 1 - \lfloor k \rfloor, \ldots, c\}$ is optimal.*

## 5. Decentralization Cost (DC)

A primary question regarding the DC is whether the centralized solution is a Nash equilibrium. If this is the case, then the centralized solution is attainable (as discussed at the end of §2), and the DC is irrelevant. It can be easily checked whether the centralized solution is an equilibrium, via a simple enumeration of each department's feasible strategies, while the opponents' strategies are held fixed at the centralized strategies. Such computations are not time consuming, even for large problems.

When the centralized solution is not an equilibrium, the DC should be considered. Computation of the exact value of the DC involves the difficult task of computing Nash equilibria. This computation is costly because the number of strategy combinations that are potential candidates as a pure equilibrium grows quickly with respect to the problem size. Reducing the set of potential strategy combinations, based on the dominance properties derived in the previous section, does not reduce the computational difficulty significantly. Furthermore, in many cases no pure equilibrium exists, leading to the need to compute mixed equilibria, involving the more difficult choice of sets of pure strategies and their respective probabilities. Therefore, instead of an exact computation, we investigate bounds for the DC that are specific to a given problem, i.e., given specific values of number of departments $n$, job durations $t_j$, and job ownership $(C_i)_i$. First, we derive an upper bound using the dominance properties derived in the previous section. This upper bound is easy to compute for any problem size. We show how this bound performs for typical distributions of problem parameters. We then derive lower and upper bounds based on correlated equilibria. Contrary to the former upper bound, these bounds suffer from costly computations because of fast growth in the number of strategy combinations. However, they are much tighter bounds, and moreover, again using the dominance properties from the previous section, we are able to compute them for problems large enough to achieve insights on the DC.

Throughout this section, $(\mathcal{M}_i^*)_i = (M_{ih_i}^*, p_{ih_i}^*)_{i,h_i}$ will denote any (mixed) Nash equilibrium. Furthermore, denote the optimal value of the firm's objective by $F^* = \min_{(\mathcal{M}_i)_i} F_0((\mathcal{M}_i)_i)$. Note also that $F^* = \min_{(M_i)_i} F_0((M_i)_i)$, because no gain is achieved by mixing optimal strategies.

### 5.1. Upper Bound

The following proposition is useful for the derivation of an upper bound for the DC. The proposition shows that the firm's equilibrium objective value is bounded from above by each of two strategy combinations, one keeping the jobs of all departments in-house and the other sending all jobs to the subcontractor. Intuitively, the relative effectiveness of these bounds depends on the value of $k$, in the sense that the former bound is effective for high values of $k$ and the latter for low values.

Proposition 3. $F_0((\mathcal{M}_i^*)_i) \leq \min\{F_0((C_i)_i), F_0((\varnothing)_i)\}$.

The proof is in the appendix. Notice that the above inequality is not obvious because an equilibrium may result in a relatively poor value in terms of the firm's objective compared to the centralized solution. This is demonstrated in Example 1, where the firm's objective at equilibrium is at least as high as any other strategy profile, except for those that determine the right-hand side of the above inequality. In contrast, when the value of $k$ is set to 2.5, the right-hand side expression equals the firm's objective at equilibrium, and both are, at most, as high as any other strategy profile except for the centralized solution.

In Theorem 1, we improve upon Proposition 3 by presenting an upper bound $B$ to the firm's decentralized objective value. This upper bound is the maximal value over all strategy combinations satisfying three conditions, following the solution properties derived in previous sections. Condition (1) states that the overall number of jobs that are processed in-house exactly equals the sum of the maximal possible values according to Proposition 1; namely, $\sum_i \min\{\lfloor k\rfloor, |C_i|\}$. Condition (2) states that within each segment, i.e., consecutive jobs belonging to a single department, only the longest jobs are processed in-house. Condition (3) requires that segments of jobs having at least one job performed in-house are consecutive within all segments. In other words, between any two such segments, there is no segment having all jobs sent to the subcontractor. The upper bound $B$ can be computed as the maximum of, at most, $c-1$ strategy combinations, so it is easy to compute. Thus, $B$ may be used to construct an upper bound to the DC. Nevertheless, there are instances for which $B$ requires too many or too few in-house jobs, thus, for such instances, the right-hand side of the inequality in Proposition 3 is strictly lower than $B$. Consequently, we define an upper bound $U$ as the minimum of these two values.

Theorem 1. *Let* $U = \min\{F_0((C_i)_i), F_0((\varnothing)_i), B\}/F^*$, *where* $B = \max F_0((M_i)_i)$ *over* $(M_i)_i$, *satisfying the following three conditions*:

(1) $|\bigcup_i M_i| = \sum_i \min\{\lfloor k\rfloor, |C_i|\}$,

(2) $\forall\, i, s, j, j'$ *such that* $j, j' \in S_i^s$ *and* $j > j'$, *if* $j' \in M_i$, *then* $j \in M_i$, *and*

(3) $\forall\, i, s, j, j', j''$ *such that* $j > j' > j''$ $j' \in S_i^s$ *and* $M_i \cap S_i^s = \varnothing$, *if* $j \in \bigcup_{i'} M_{i'}$, *then* $j'' \notin \bigcup_{i'} M_{i'}$ *and if* $j'' \in \bigcup_{i'} M_{i'}$, *then* $j \notin \bigcup_{i'} M_{i'}$.

*Then* $U$ *is an upper bound for* $F_0((\mathcal{M}_i^*)_i)/F^*$.

Example 2. Recall Example 1, where $c = 3$, $n = 2$, and in-house jobs durations $t_j$ are 1, 2, 5, $C_1 = \{1, 3\}$, $C_2 = \{2\}$, and the subcontractor offers $k = 1.5$. To compute the upper bound for this example, in both cases where all jobs are processed in-house and where all jobs are sent to the subcontractor, the firm's objective value is $9 + 3 = 12$. The bound $B = \max\{F_0(\{1\}, \{2\}), F_0(\{3\}, \{2\})\} = \max\{8.5 + 3, 8.5 + 2\} = 11.5$. Therefore, the upper bound is $\min\{12, 11.5\}/9.5 \simeq 1.211$, which is exactly the DC of one of the equilibrium points discussed in Example 1.

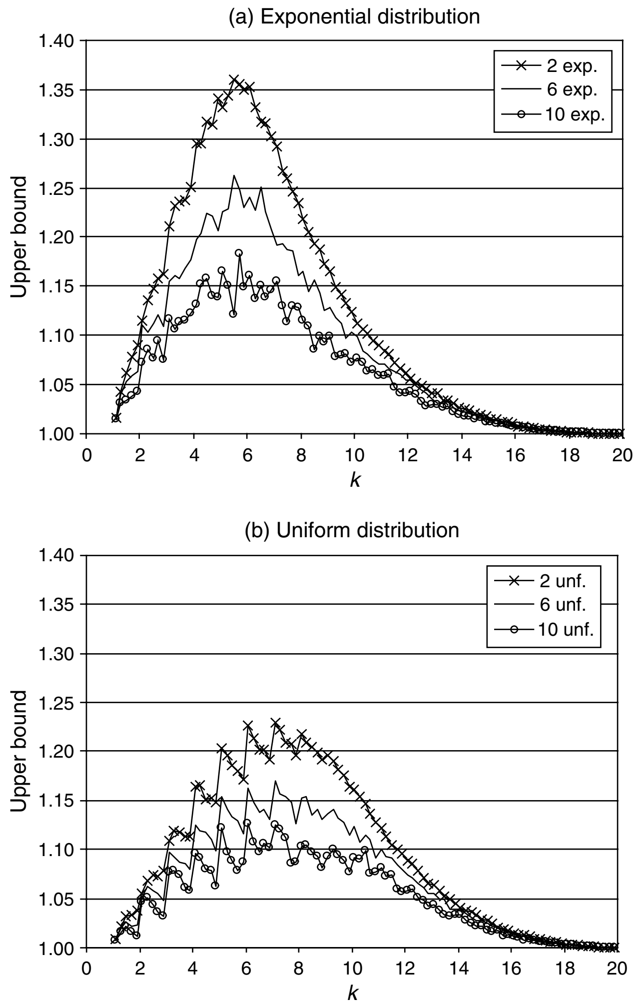
The following experimentation demonstrates the behavior of the upper bound, $U$, for common production situations. We demonstrate empirically how the problem characteristics affect the value of the upper bound.

A set of 20-job problems is considered, where the value of $k$ changes between 1 and 20. Two types of job durations are generated: the first based on a uniform distribution and the second on an exponential distribution, while the expected value is kept the same for both distributions for comparison purposes. The jobs are divided between two departments, while considering three alternative values of expected size of segments of consecutive jobs belonging to a single department: 2, 6, and 10. Segment sizes are determined by altering job ownership using a geometric distribution. For each combination of $k$ and expected segment size, the upper bound was calculated for 100 randomly generated problems.

Results associated with the uniform and exponential distributions are demonstrated in Figures 1(a) and 1(b), respectively. The following observations are identified.

Observation 1. The average value of the upper bound decreases with the expected segment size.

**Figure 1** Upper Bound for Various Segment Sizes/Job Duration Distributions




Observation 2. The upper bound associated with the exponential distribution gives, in general, higher values than the one associated with the uniform distribution.

For Observation 1, intuitively, the smaller the segment size, the higher the incentive to process shorter jobs in-house in order to advance in queue, thus increasing the DC by moving away from the centralized optimal decision to process only the longest jobs in-house.

### 5.2. Correlated Equilibrium Bounds

An alternative approach for developing bounds for the DC is based on correlated equilibria. In a correlated equilibrium (Aumann 1974), the DMs choose correlated strategies, i.e., they follow a probability distribution $p((M_i)_i)$ over pure-strategy profiles. Each DM's choice minimizes the expected cost, calculated over all possible strategy profiles that other DMs may choose. This is formulated by the constraints of the following LP program. Recall that $(M_i, M_{-i})$ as well as $(M_i)_i$ denote a pure-strategy profile for all players.

$$\min \text{ or } \max_{p((M_i)_i)} \sum_{(M_i)_i} p((M_i)_i) F_0((M_i)_i)$$

$$\text{s.t.} \quad \sum_{M_{-i}} p(M_i, M_{-i})[F_i(M_i, M_{-i}) - F_i(\widehat{M}_i, M_{-i})] \leq 0 \qquad \forall\, i,\ \forall\, M_i,\ \widehat{M}_i \subseteq C_i \tag{LP}$$

$$\sum_{(M_i)_i} p((M_i)_i) = 1$$

$$0 \leq p((M_i)_i) \leq 1 \quad \forall\, (M_i)_i$$

The first set of constraints requires that department $i$, whenever it is supposed to choose the pure strategy $M_i$, does not benefit from deviating to any other pure strategy $\widehat{M}_i$. The remaining constraints require that $p((M_i)_i)$ form a probability distribution over pure-strategy profiles. Notice that the probabilities $p((M_i)_i)$ are the only decision variables. These probabilities replace the mixed-strategy joint probabilities $\prod_i p_{ih_i}$, which assume independence between the marginal probabilities $p_{ih_i}$ that the departments assign to their own pure strategies. Unlike mixed Nash equilibria, correlated equilibria may have interdependencies between these marginal probabilities. Thus, any pure/mixed Nash equilibrium is a correlated equilibrium, but not vice versa. Applying this fact to our problem, the best- and worst-correlated equilibria in terms of the firm's objective provide lower and upper bounds for pure/mixed Nash equilibria. This is exactly the purpose of the objective function in the LP program above: minimize (maximize) the firms objective in equilibrium to provide lower (upper) bounds for the DC.

The following experimentation demonstrates the behavior of the correlated equilibrium lower and upper bounds, thus allowing some insights on the DC. A set of 200 instances is considered, where the number of jobs $c$ randomly changes between 5 and 20. The value of $k$ randomly changes between 1 and $c$, where $1 \leq k \leq c/2$ for the first 100 instances and $c/2 \leq k \leq c$ for all other instances. Job durations are uniformly distributed between 2 and 10. Segment sizes are determined by altering job ownership between two departments using a geometric distribution with a random parameter (e.g., if a parameter of 0.8 was generated for a certain instance, then with probability 0.8, the next job ownership is identical to the previous one). Both the lower and upper bounds were computed for each of 200 instances using the LP program above. The dominance properties derived in the previous section were helpfully implemented in reducing the number of strategy combinations, thus allowing the computation for up to 20 jobs.

Hereafter, we report the computed bounds for the DC in percentage values (i.e., after subtracting 1 from the bound and then multiplying by 100). The experiments result in average lower- and upper-bound values of 3.6% and 3.7%, respectively, with a maximum value of 10% for both. The following observations elaborate on this result.

Observation 3. An average (maximal) difference of 0.1% (6.72%) was obtained between the lower and upper bounds.

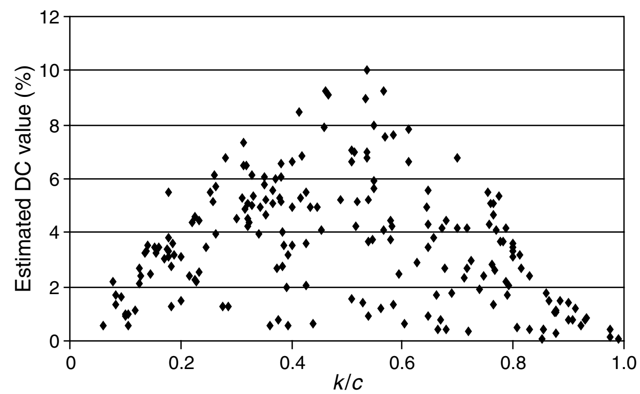
This observation indicates that the correlated equilibrium lower and upper bounds for the DC are tight. In most of the instances, a unique objective value was obtained at pure/mixed Nash equilibrium, while a very small difference was obtained between the lower and upper bounds in the remaining instances.

For the next observation, denote by $\lambda$ the total number of segments for all departments, $\sum_i l_i$.

Observation 4. The DC increases for $k/c \in [0, \frac{1}{2}]$ and decreases for $k/c \in [\frac{1}{2}, 1]$. Moreover, the DC increases with $\lambda/c$.

This observation is confirmed by regressing the DC against $k/c$ and $\lambda/c$, in which case both explanatory variables come out significant with $p$-values less than 0.01. Intuitively, each department's best strategy is less dependent on other departments for small or large values of $k/c$. In the former, the department prefers sending most jobs to the subcontractor, while in the latter, processing most jobs in-house is preferred. The conflict is more evident for midrange values of $k/c$, which thus results in a relatively high DC. These results are well demonstrated in Figure 2 (in light of Observation 3, the DC value can be estimated based on either the lower or upper bounds, with no significant change in the estimate). Note that

**Figure 2 Estimated DC Value (in Percentage) as a Function of $k/c$**




the range of DC values for each value of $k/c$ results from the range of values for $\lambda/c$.

Observation 5. A pure Nash equilibrium was obtained in 134 out of 200 instances (67%). Moreover, for $k/c \le \frac{1}{2}$, only 36 out of 100 instances (36%) were observed with a pure Nash equilibrium, while for the remaining instances with $k/c > \frac{1}{2}$, 96 out of 100 (96%) were identified with a pure Nash equilibrium.

The intuitive explanation for the lack of pure Nash equilibrium for low values of $k$ boils down to a choice cycle, in which departments tend to prefer the subcontractor, thus reducing the queue on the in-house machine, which in turn reduces the incentive to use the subcontractor. Such a cycle precludes pure Nash equilibria (see Table 2 in Example 1). In such cases, there always exists a mixed equilibrium. For high values of $k$, the incentive to use the in-house machine is sufficiently intense to break down the cycle, thus yielding a pure Nash equilibrium.

The robustness of the above observations was demonstrated by further experimentation of 30 instances with 3 to 5 departments.

Observation 6. An average (maximal) difference of 0.3% (2.67%) was obtained between the lower and upper correlated equilibrium bounds, indicating the tightness of both bounds. Moreover, behavior similar to the one shown in Figure 2 was identified and a pure Nash equilibrium was found for 20 out of the 30 instances (8 out of 18 instances with $k/c \le \frac{1}{2}$ and 12 out of 12 instances with $k/c > \frac{1}{2}$).

Although Observation 3 indicates that the correlated equilibrium bounds are tight, and in most cases provide the value of the DC, these bounds are hard to calculate, in contrast to the upper bound $U$. Nevertheless, the correlated equilibrium bounds enable us to evaluate the quality of $U$. To this end, the following experimentation compares the correlated equilibrium lower bound and the upper bound $U$. A set of 100, 20-job instances is considered, where the value of $k$ randomly changes between 1 and 20 and segment sizes are randomly determined by altering job ownership between two departments. For each instance, the ratio of the upper bound $U$ and the correlated equilibrium lower bound is calculated. This ratio is greater than or equal to $U$ divided by the DC (equality is obtained when the correlated equilibrium lower bound is equal to the DC).

Observation 7. The average value of the ratio between the two bounds is 4.6%. As expected, higher values can obtain around $k = c/2$. For 47 observations (out of 100), there was $5 \leq k \leq 15$, and the average value of the ratio for these instances was 7.4%.

Based on these results, along with Observation 3, the correlated equilibrium bounds should be used when computationally tractable, and in other cases, the upper bound $U$ can be used as a reasonable substitute.

## 6. Scheduling-Based Coordinating Mechanism

When the centralized solution is not an equilibrium, a mechanism that coordinates the decentralized scheduling environment should be considered. Clearly, if the potential savings associated with the bounds for the DC are lower than the cost of implementing such a mechanism, then it should not be implemented.

A coordinating mechanism should change the incentive schemes of the departments, as derived from their scheduling objectives, so that all prefer to act according to the centralized objective. As explained in the introduction, two main factors determine the applicability of such a coordinating mechanism in general. First, it should be designed based on information available to the general director, who is typically less informed than the departments. Second, to achieve efficiency under the modified incentives, internal money transfers within the firm should be noncostly, i.e., always add up to zero. The mechanism proposed below satisfies both requirements. In particular, only limited information is required about actual dispatching decisions, with no need to know each job's duration and ownership. Moreover, there are no money transfers involved in the mechanism, and instead coordination is achieved merely by modifying DM 0's SPT scheduling policy.

**Table 3** **Centralized Nash Equilibrium Under IC-SPT**

| $M_1\backslash M_2$ | $\varnothing$ | $\{2\}$ |
|---|---|---|
| $\varnothing$ | (9, 3) | (9, 2) |
| $\{1\}$ | (8.5, 3) | (10.5, 2) |
| $\{3\}$ | **(6.5, 3)** | (6.5, 7) |
| $\{1, 3\}$ | (11, 3) | (15, 2) |

The proposed mechanism modifies the SPT rule used for in-house jobs, so that departments prefer not to deviate from the centralized solution. The mechanism requires that all in-house jobs other than the shortest are processed according to the SPT rule. The shortest in-house job is processed last if more than $\lfloor k \rfloor$ jobs are processed in-house or if the largest job sent to the subcontractor is longer than the shortest in-house job. Otherwise, the shortest in-house job is processed first. We next provide the definition of the mechanism and then show, in the subsequent theorem, that it indeed coordinates the system.

For any $C' \subseteq C$, let $j_S(C')$, $j_L(C')$ be the shortest and longest jobs in $C'$, respectively.

Definition 2 IC-SPT (Incentive Compatible SPT). For any $(M_i)_i$, all jobs $j \in (\bigcup_i M_i)\backslash j_S(\bigcup_i M_i)$ are processed according to the SPT rule. Furthermore, if $|\bigcup_i M_i| > \lfloor k \rfloor$ or $t_{j_L(C\backslash \cup_i M_i)} > t_{j_S(\cup_i M_i)}$, then $j_S(\bigcup_i M_i)$ is processed in-house last; otherwise, it is processed first.

The sum of completion times of $i$'s jobs under IC-SPT, denoted by $F_i^{\mathrm{IC}}(M_i, M_{-i})$, is determined by adapting Equation (1) to this scheduling rule.

Theorem 2. *Under IC-SPT, there exists a centralized solution pure Nash equilibrium* $(M_i^{\mathrm{IC}})_i$, *where* $M_i^{\mathrm{IC}} = C_i \cap \{c+1-\lfloor k \rfloor, \dots, c\}$.

Example 3. Consider again Example 1, where there are two departments and three jobs with in-house durations 1, 2, 5, $C_1 = \{1, 3\}$, $C_2 = \{2\}$, and the subcontractor offers $k = 1.5$. Implementing the mechanism IC-SPT, the sum of completion times for each pure-strategy combination is given in Table 3.

The mechanism alters only all strategy combinations with more than one job processed in-house. For example, when $M_1 = \{1\}$ and $M_2 = \{2\}$, because $|\bigcup_i M_i| > \lfloor k \rfloor$, the longer job 2 is processed before job 1; thus, $F_1^{\mathrm{IC}}(M_1, M_2) = (1+2) + (1.5 \cdot 5) = 10.5$ and $F_2^{\mathrm{IC}}(M_1, M_2) = 2$. The sum of completion times of the unique pure Nash equilibrium is marked in Table 3 with bold letters. In this equilibrium, only the longest job is processed in-house, which is exactly the centralized solution when $\lfloor k \rfloor = 1$.

## 7. Summary

Competition over limited resources, along with scheduling problems, often cause inefficiency in organizations that operate under decentralized decision processes. This occurs when several DMs, each interested in optimizing individual performance measures, must process jobs using common resources. Moreover, it is possible in such circumstances that DMs neither cooperate nor have incentives to act according to a centralized optimum. Thus, a question arises whether it is worthwhile to spend resources to achieve a centralized decision to overcome decentralized inefficiency.

In this paper, we evaluated the loss mentioned above by analyzing the DC, i.e., the ratio between the total cost in pure/mixed Nash equilibrium when DMs are involved in a noncooperative game and the total cost in the centralized optimum. We developed a general approach for decentralized modelling, in which a coordinating mechanism is implemented when the centralized solution is not a Nash equilibrium and the cost of implementing the mechanism is justified given the DC.

We analyzed a dispatching-sequencing model in which each DM must decide whether to process each job on a resource with limited capacity (e.g., in-house common resource) or on a less efficient resource with unlimited capacity (e.g., subcontractor). Examples demonstrate that the DC is not negligible in this model. We characterized the general centralized optimum analytically, but computation of the decentralized solution proved more difficult. Instead, bounds were presented on the number of jobs processed on the efficient common resource in-house and were used to derive bounds for the DC. We presented upper bounds that are easily computable for problems with both an arbitrary number of jobs and an arbitrary number of DMs. We also presented tight, but harder to compute, lower and upper bounds based on correlated equilibria. The bounds for the DC were illustrated for typical distributions of problem parameters.

Finally, a scheduling-based coordinating mechanism was proposed for cases where the centralized solution is not an equilibrium and the potential savings associated with the bounds for the DC are not lower than the cost of implementing such a mechanism. This mechanism modifies the scheduling rule used for in-house jobs, so that departments prefer not to deviate from the centralized solution.

We view our approach as useful for general decentralized scheduling situations and hope to apply it in further research of scheduling games.

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### Appendix

Proof of Proposition 1. (1) Suppose that for some $h_i$, $|M_{ih_i}| > \lfloor k \rfloor$. We show that irrespective of the strategies of $i$'s opponents, it is always worthwhile for $i$ to send the shortest job currently in-house to the subcontractor. Note that $|M_{ih_i}| > k$ because $|M_{ih_i}| \geq \lfloor k \rfloor + 1 > k$. Let $j \in M_{ih_i}$ such that $t_j = t_{M_{ih_i}}(1)$ and let $M'_{ih_i} \equiv M_{ih_i} \backslash \{j\}$. Let $\mathcal{M}'_i$ equal $\mathcal{M}_i$ except that $M_{ih_i}$ is replaced by $M'_{ih_i}$. Let $\mathcal{M}_{-i}$ be any strategy profile for the opponents. Then, by Equation (2), $F_i(\mathcal{M}_i, \mathcal{M}_{-i}) - F_i(\mathcal{M}'_i, \mathcal{M}_{-i}) = \sum_{(h_{i'})_{i'}} p_{ih_i} \prod_{i' \neq i} p_{i'h_{i'}} [F_i(M_{ih_i}, (M_{i'h_{i'}})_{i' \neq i}) - F_i(M'_{ih_i}, (M_{i'h_{i'}})_{i' \neq i})]$. Within this expression, all terms without $h_i$ cancel out. Moreover, omitting the second term of Equation (1) lowers the difference, so $F_i(\mathcal{M}_i, \mathcal{M}_{-i}) - F_i(\mathcal{M}'_i, \mathcal{M}_{-i}) \geq p_{ih_i} \sum_{(h_{i'})_{i' \neq i}} \prod_{i' \neq i} p_{i'h_{i'}}$ $[|M_{ih_i}| t_j - kt_j] > 0$. Thus, $\mathcal{M}_i$ is strictly dominated by $\mathcal{M}'_i$, because the latter strict inequality holds for any $\mathcal{M}_{-i}$.

(2) Suppose that $|\bigcup_i M_i^*| < \lfloor k \rfloor - 1$. We show that it is always worthwhile to perform in-house at least one job that is currently sent to the subcontractor. Note that $\bigcup_i M_i^* \neq C$ because $|\bigcup_i M_i^*| < \lfloor k \rfloor \leq k < c$, so there exists some $i'$ and $j' \in C_{i'} \backslash M_{i'}^*$. Let $M'_{i'} \equiv M_{i'}^* \cup \{j'\}$ and let $1 \leq \overline{m} \leq |M'_{i'}|$ such that $t_{M'_{i'}}(\overline{m}) = t_{j'}$. Thus, $k \geq \lfloor k \rfloor > |(\bigcup_i M_i^*)| + 1 = |(\bigcup M_{-i'}^*) \cup M'_{i'}|$. Therefore, by Equation (1), $F_{i'}(M'_{i'}, M_{-i'}^*) - F_{i'}(M_{i'}^*, M_{-i'}^*) = (|M'_{i'}| + 1 - \overline{m}) t_{j'} + \sum_{\{j \in \cup_i M_i^* \mid j' > j\}} t_j - kt_{j'} \leq |(\bigcup M_{-i}^*) \cup M'_{i'}| t_{j'} - kt_{j'} < 0$, and so $M_{i'}^*$ is not a best response to $M_{-i'}^*$. Therefore $|\bigcup_i M_i^*| \geq \lfloor k \rfloor - 1$.

If $k > \lfloor k \rfloor$ and we assume that $|\bigcup_i M_i^*| = \lfloor k \rfloor - 1$, then using similar arguments, for some $i'$, $M_{i'}^*$ is not a best response to $M_{-i'}^*$, concluding that if $k > \lfloor k \rfloor$, then $|\bigcup_i M_i^*| \geq \lfloor k \rfloor$. If $k = \lfloor k \rfloor$, then $F_{i'}(M'_{i'}, M_{-i'}^*) \leq F_{i'}(M_{i'}^*, M_{-i'}^*)$ and, in fact, must hold as an equality because $(M_i^*)_i$ is an equilibrium; thus, $(M_i^{**})_i$ can be reached from $(M_i^*)_i$ by replacing $M_{i'}^*$ by $M'_{i'}$. □

Proof of Proposition 2. Without loss of generality, we can assume the following two conditions: (1) $\{j'\} \cup \{j+1, \ldots, \max\{j \mid j \in S_i^s\}\} \subseteq M_{ih_i}$, but $j \notin M_{ih_i}$ and (2) $j'' \notin M_{ih_i}$

for any $j'' \in S_i^s$ such that $j' < j'' \le j$. Let $\bar{m} = |\{j'' \in M_{ih_i} \mid j'' \ge j'\}|$. By Proposition 1, $\bar{m} \le |M_{ih_i}| \le \lfloor k \rfloor$. Let $M'_{ih_i} \equiv M_{ih_i} \cup \{j\} \backslash \{j'\}$ and let $\mathscr{M}'_i$ equal $\mathscr{M}_i$ except that $M_{ih_i}$ is replaced by $M'_{ih_i}$. Let $\mathscr{M}_{-i}$ be any strategy profile for the opponents. Then, $F_i(\mathscr{M}'_i, \mathscr{M}_{-i}) - F_i(\mathscr{M}_i, \mathscr{M}_{-i}) = \sum_{(h_{i'})_{i'}} \prod_{i'} p_{i'h_{i'}} [F_i(M'_{ih_i}, (M_{i'h_{i'}})_{i' \neq i}) - F_i(M_{ih_i}, (M_{i'h_{i'}})_{i' \neq i})] = \sum_{(h_{i'})_{i'}} \prod_{i'} p_{i'h_{i'}} (\bar{m} - k)(t_j - t_{j'}) \le 0$ because $t_j > t_{j'}$ and $\bar{m} \le \lfloor k \rfloor \le k$. Thus, $F_i(\mathscr{M}'_i, \mathscr{M}_{-i}) \le F_i(\mathscr{M}_i, \mathscr{M}_{-i})$ for any $\mathscr{M}_{-i}$, so $\mathscr{M}_i$ is dominated by $\mathscr{M}'_i$. Notice that the latter inequality holds as an equality only when $\bar{m} = \lfloor k \rfloor = k$, i.e., when $k$ is an integer and $i$ has exactly $\lfloor k \rfloor$ jobs above, and including $j'$ processed in-house. $\square$

Proof of Corollary 1. The proof follows directly from Propositions 1 and 2 in the case $n = 1$, because the objective of the single player is identical to the centralized objective, and finding a pure best response is equivalent to finding an optimal strategy. $\square$

Proof of Proposition 3. Note that for each department $i$, Equation (1) implies that for each $h_i$, $F_i(C_i, C_{-i}) \ge F_i(C_i, (M^*_{i'h_{i'}})_{i' \neq i})$, because $\bigcup (M^*_{i'h_{i'}})_{i' \neq i} \subseteq \bigcup C_{-i}$ and the completion times are at least as high when all, rather than some, of the other departments' jobs are processed in-house. Thus, for each $i$, $F_i(C_i, C_{-i}) = \sum_{(h_{i'})_{i'}} \prod_{i'} p^*_{i'h_{i'}} F_i(C_i, C_{-i}) \ge \sum_{(h_{i'})_{i'}} \prod_{i'} p^*_{i'h_{i'}} F_i(C_i, (M^*_{i'h_{i'}})_{i' \neq i}) = F_i(C_i, \mathscr{M}^*_{-i})$. Therefore, $F_i(\mathscr{M}^*_i, \mathscr{M}^*_{-i}) - F_i(C_i, C_{-i}) \le F_i(\mathscr{M}^*_i, \mathscr{M}^*_{-i}) - F_i(C_i, \mathscr{M}^*_{-i}) \le 0$, where the last inequality follows because $((\mathscr{M}^*_i)_i)$ is an equilibrium. It follows that $F_0((\mathscr{M}^*_i)_i) - F_0((C_i)_i) = \sum_i [F_i(\mathscr{M}^*_i, \mathscr{M}^*_{-i}) - F_i(C_i, C_{-i})] \le 0$, thus establishing half of the proof. Similarly, for the other half, $F_i(\varnothing, \varnothing) = F_i(\varnothing, \mathscr{M}^*_{-i})$; thus, $F_0((\mathscr{M}^*_i)_i) - F_0((\varnothing)_i) = \sum_i [F_i(\mathscr{M}^*_i, \mathscr{M}^*_{-i}) - F_i(\varnothing, \varnothing)] = \sum_i [F_i(\mathscr{M}^*_i, \mathscr{M}^*_{-i}) - F_i(\varnothing, \mathscr{M}^*_{-i})] \le 0$. $\square$

Proof of Theorem 1. By Proposition 2, there exists $(\mathscr{M}^{**}_i)_i = (M^{**}_{ih_i}, p^*_{ih_i})_{i, h_i}$ such that $F_0((\mathscr{M}^{**}_i)_i) = F_0((\mathscr{M}^*_i)_i)$, and for all $(h_i)_i$, $(M^*_{ih_i})_i$ satisfies Condition (2). Thus, by Proposition 3, it is sufficient to show that $B \ge F_0((\mathscr{M}^{**}_i)_i)$. By Proposition 1, $|M^{**}_{ih_i}| \le \min\{\lfloor k \rfloor, |C_i|\}$. Let $(\mathscr{M}'_i)_i = (M'_{ih_i}, p^*_{ih_i})_{i, h_i}$ such that for all $i$, $h_i$, $M'_{ih_i} \supseteq M^{**}_{ih_i}$, $|M'_{ih_i}| = \min\{\lfloor k \rfloor, |C_i|\}$, and Condition (2) is maintained. We first show that $F_0((\mathscr{M}'_i)_i) \ge F_0((\mathscr{M}^{**}_i)_i)$. Similar to the proof of Proposition 3, for each $i$ and $(h_i)_i$, Equation (1) implies that $F_i(M'_{ih_i}, (M'_{i'h_{i'}})_{i' \neq i}) \ge F_i(M'_{ih_i}, (M^{**}_{i'h_{i'}})_{i' \neq i})$ because $\bigcup (M^{**}_{i'h_{i'}})_{i' \neq i} \subseteq \bigcup (M'_{i'h_{i'}})_{i' \neq i}$ and because the completion times are at least as high when all, rather than some, of the other department's jobs are processed in-house. Thus, for each $i$, $F_i(\mathscr{M}'_i, \mathscr{M}'_{-i}) = \sum_{(h_{i'})_{i'}} \prod_{i'} p^*_{i'h_{i'}} F_i(M'_{ih_i}, (M'_{i'h_{i'}})_{i' \neq i}) \ge \sum_{(h_{i'})_{i'}} \prod_{i'} p^*_{i'h_{i'}} F_i(M'_{ih_i}, (M^{**}_{i'h_{i'}})_{i' \neq i})] = F_i(\mathscr{M}'_i, \mathscr{M}^{**}_{-i})$. Therefore, $F_i(\mathscr{M}^{**}_i, \mathscr{M}^{**}_{-i}) - F_i(\mathscr{M}'_i, \mathscr{M}'_{-i}) \le F_i(\mathscr{M}^{**}_i, \mathscr{M}^{**}_{-i}) - F_i(\mathscr{M}'_i, \mathscr{M}^{**}_{-i}) \le 0$, where the last inequality follows because $((\mathscr{M}^{**}_i)_i)$ is an equilibrium. It follows that $F_0((\mathscr{M}^{**}_i)_i) - F_0((\mathscr{M}'_i)_i) = \sum_i [F_i(\mathscr{M}^{**}_i, \mathscr{M}^{**}_{-i}) - F_i(\mathscr{M}'_i, \mathscr{M}'_{-i})] \le 0$, as required. It is now sufficient to show that for all $(h_i)_i$, $B \ge F_0((M'_{ih_i})_i)$, because this would imply that $B \ge F_0((\mathscr{M}'_i)_i)$. To this end, fix $(h_i)_i$. Recall that $(M'_{ih_i})_i$ satisfies Conditions (1) and (2), but not necessarily Condition (3). Let $(M''_{ih_i})_i$ such that $|\bigcup_i M''_{ih_i}| = |\bigcup_i M'_{ih_i}|$, for all $m \le |\bigcup_i M'_{ih_i}| - \lfloor k \rfloor$, $t_{\cup_i M''_{ih_i}}(m) \ge t_{\cup_i M'_{ih_i}}(m)$, for all $m > |\bigcup_i M'_{ih_i}| - \lfloor k \rfloor$, $t_{\cup_i M''_{ih_i}}(m) \le t_{\cup_i M'_{ih_i}}(m)$, and $(M''_{ih_i})_i$ satisfies Conditions (1)–(3). By Equation (3), $F_0((M''_{ih_i})_i) - F_0((M'_{ih_i})_i) = \sum_{m=1}^{|\cup_i M'_{ih_i}|} (|\bigcup_i M'_{ih_i}| + 1 - m - k)[t_{\cup_i M''_{ih_i}}(m) - t_{\cup_i M'_{ih_i}}(m)]$. Each term in this sum is nonnegative, because either $m \le |\bigcup_i M'_{ih_i}| - \lfloor k \rfloor \le |\bigcup_i M'_{ih_i}| + 1 - k$ and $t_{\cup_i M''_{ih_i}}(m) \ge t_{\cup_i M'_{ih_i}}(m)$, or $m - 1 \ge |\bigcup_i M'_{ih_i}| - \lfloor k \rfloor \ge |\bigcup_i M'_{ih_i}| - k$ and $t_{\cup_i M''_{ih_i}}(m) \le t_{\cup_i M'_{ih_i}}(m)$. Thus, $B \ge F_0((M''_{ih_i})_i) \ge F_0((M'_{ih_i})_i)$ as required. $\square$

Proof of Theorem 2. It is sufficient to show that for any $i$, $M_i^{IC}$ is a best response to $M_{-i}^{IC}$. Fix $i$ and let $M_i \subseteq C_i$ and $C' = (\bigcup M_{-i}^{IC}) \cup M_i$. Suppose first that $|C'| < \lfloor k \rfloor$; then there exists $j' \in C_i \backslash M_i$, because $\lfloor k \rfloor < c$ and only $i$ deviates from processing in-house $\lfloor k \rfloor$ jobs. Let $M'_i = M_i \cup \{j'\}$, then $F_i^{IC}(M'_i, M_{-i}^{IC}) - F_i^{IC}(M_i, M_{-i}^{IC}) \le (|C'| - k)t_{j'} < (\lfloor k \rfloor - k)t_{j'} \le 0$; thus, $M_i$ is not a best response to $M_{-i}^{IC}$. Now suppose that $|C'| > \lfloor k \rfloor$ or ($|C'| = \lfloor k \rfloor$ and $t_{j_L(C \backslash C')} > t_{j_S(C')}$). Then $j_S(C') \in C_i$, because only $i$ deviates from processing the $\lfloor k \rfloor$ longest jobs in-house. Thus, $j_S(C')$ is processed last under IC-SPT. Let $M'_i = M_i \backslash \{j_S(C')\}$; then $F_i^{IC}(M'_i, M_{-i}^{IC}) - F_i^{IC}(M_i, M_{-i}^{IC}) = k t_{j_S(C')} - \sum_{j \in C'} t_j < (k - |C'|) t_{j_S(C')} \le 0$. Thus, again, $M_i$ is not a best response to $M_{-i}^{IC}$. Because there always exists a best response to $M_{-i}^{IC}$ and only $M_i^{IC}$ was not ruled out, we conclude that $M_i^{IC}$ is a best response to $M_{-i}^{IC}$. $\square$

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