Subgame Perfect Farsighted Stability*

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Abstract

Farsighted decision makers, who anticipate that their deviations from a given course of action may lead to deviations by others, act differently from myopic decision makers. In this paper we propose a new farsighted approach to strategic interactions settings, referred to as the Subgame Perfect Consistent Set (SPCS), based on consistency in the spirit of the von Neumann Morgenstern solution and on subgame perfect equilibrium. Rather than follow constructs such as indirect dominance, farsighted players according to the SPCS adopt best responses, and unlike expectation function-based farsighted solution concepts, the SPCS incorporates explicitly inherent uncertainties in the abstract game model. We show the SPCS exists for any finite game. Surprisingly, the SPCS is shown to always lead to Pareto efficiency in farsighted normal form games. This result is demonstrated in various oligopolistic settings, and is shown to imply, for example, that players who follow the SPCS reasoning are always able to share the monopolistic profit in farsighted settings based on Bertrand and Cournot competition, and are always able to achieve coordination and Pareto efficiency in decentralized supply chain contracting and network formation, even when they cannot form coalitions.

Key Words: Dynamic games, normal form games, abstract games, farsighted stability, vNM consistency, oligopolistic competition

1 Introduction

The introduction of farsighted game theoretic solution concepts was motivated by criticism of 'myopia' raised against both cooperative and noncooperative approaches (Chamberlin

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1933, Harsanyi 1974, Chwe 1994). In strategic settings, it is arguably better for decision makers to conduct their actions based on farsighted considerations, and to learn how to be farsighted. As compared to 'myopic' players who follow the vNM solution, core, or Nash equilibrium, farsighted players recognize that their own moves may induce multiple moves by other players. Motivated by the shortcoming of myopic reasoning, a variety of farsighted solutions were proposed, starting with Harsanyi (1974). He pioneered the approach that coalitional behavior should be analyzed using subgame perfect equilibrium strategies of an extensive form game, with the aim of providing a non-cooperative foundation to the vNM solution in cooperative game theory.

In view of the 'myopia' of the vNM solution, Harsanyi (1974) has further suggested the need to replace therein direct dominance with indirect dominance. Indeed, in the Largest Consistent set (LCS) introduced by Chwe (1994) for abstract games, farsightedness is embodied by indirect dominance.[1] Chwe (1994) proved that the LCS exists, and that the vNM solution for abstract games, wherein the direct dominance relation is replaced by indirect dominance and referred to as vNM farsighted stable set (vNM FSS), is contained in the LCS. The vNM FSS is criticized for assuming optimism on the part of the moving coalition. On the other hand, the LCS is criticized for assuming pessimism on the part of a potentially moving coalition (Ray and Vohra 2015a), and that, as a result, it could be too inclusive and may contain non-intuitive outcomes. Its conservative criterion for a move is somewhat mollified in the Largest Cautious Consistent Set (LCCS) introduced by Mauleon and Vannetelbosch (M&V, 2004).

The farsighted approach for coalitional games, derived from the classical vNM solution wherein indirect dominance replaces direct dominance, is referred to as the Harsanyi set. It was refined to address issues such as lack of "coalitional sovereignty" (Ray and Vohra 2015b), and lack of maximality in the sense of choosing better rather than best moves (Ray and Vohra 2015a, 2019). The latter has led more recently to the development of more satisfactory, due to their maximality property, farsighted solution concepts for abstract games, by relying on Rational Expectations (RE) in the sense of commonly held, endogenously determined beliefs about the continuation path following any coalition move, see, Dutta and Vohra (D&V, 2017), Dutta and Vartiainen (D&V, 2020), Kimya (2020) and Karos and Robles (K&R, 2021). These solution concepts also satisfy variants of internal and external stability in the spirit of those embodied in the vNM solution.

The RE approach relies on a deterministic expectation function. As such, it identifies, with certainty, the moving coalition, if any, at each state and the resulting new state. Any moving coalition knows with certainty the final outcome it would reach following its move

[1] For a definition of an abstract game and indirect dominance see Sections 2 and 4, respectively.

or stay at the current state. Thus, the RE approach does not incorporate inherent uncertainties in the corresponding abstract game model. Indeed, in some sense, the underlying abstract game in this approach differs from Chwe's original abstract game model, wherein, for example, it is implicitly assumed that the identity of the moving coalition at each state is generated by some unspecified and uncertain mechanism. In fact, we note that while the LCS, LCCS, or vNM FSS are criticized for employing non-optimal criteria for moves, we demonstrate in this paper that the failure of the RE approach to incorporate the inherent uncertainty as to the identity of the moving coalition may also lead to the prescription of non-optimal moves. For example, the RE approach allows that coalitions stay at a state z, even though the inherent uncerainty following a move therefrom would generate a set of final outcomes, each of which being at least as good as the outcome at state z, and some of which strictly better for all members of the moving coalition. Or the RE approach may suggest moves by coalitions from a state z, when the inherent uncertainty following that move generates a set of final outcomes, each of which being at most as good as the outcome at z, and some of which strictly worse for all members of the moving coalition.

Relatedly, a necessary implication of the RE approach is what we call the *confidence assumption*: only states predicted with certainty by the expectation function are considered by potentially moving coalitions from an initial state. This may be viewed as not fully incorporating the spirit of vNM stability, as both internal and external stability of the vNM solution, when testing an outcome with respect to a given stable set, do not exclude from consideration any outcome in the stable set.[2] One would therefore like to see *all* farsightedly stable outcomes being expected by potentially moving coalitions from an initial state.

In this paper we introduce a new farsighted solution concept for abstract games, referred to as the *Subgame Perfect Consistent Set* (SPCS). It differs from existing solution concepts in the following three respects, each of which contributing to the extension of farsighted stability beyond the confidence assumption. First, the SPCS incorporates the inherent uncertainty in the abstract game model as to the identity of the moving coalition by explicitly allowing for multiple coaltions affecting a move whenever possible. Second, players or coalitions look arbitrarily far ahead when they consider the consequences of moving or staying at the current state. From a given initial state, we model the (non-stationary) evolution of play resulting from players or coalitions' moves as an extensive form game, infinite in size. Accordingly, the SPCS uses the reasoning of subgame perfection, which endows players with unlimited farsightedness, instead of constrained farsighted constructs such as indirect

[2] Indeed, as noted by Dutta and Vohra (2017), their internal stability is weaker than the ordinary vNM internal stability since it requires internal stability only with respect to those farsighted objections that are consistent with the common expectation function. For the same reason, their external stability is stronger than the ordinary vNM external stability.

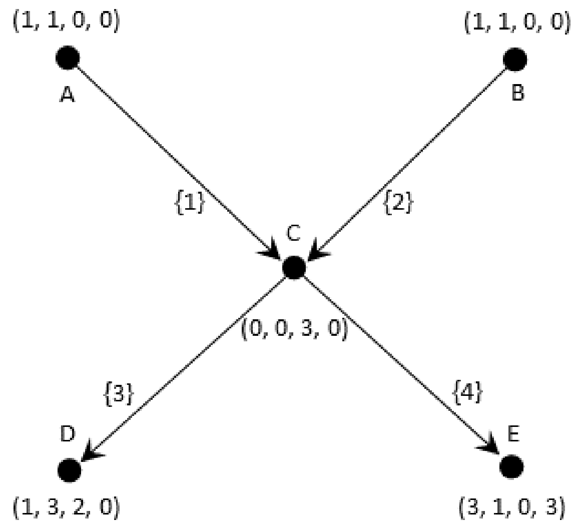

Figure 1.1: Transition tree in Example 1.1

dominance which could lead to non-optimal moves. Third and finally, we define the SPCS to consist of all states which can be supported as stable by a subgame perfect equilibrium, refined to additionally satisfy both internal and external consistency in the spirit of the vNM solution, whereby all stable states that are reachable by some continuation subgame perfect equilibrium are expected by potentially moving coalitions from an initial state. Importantly, the consideration of all reachable stable states, and not only a unique stable state reachable via indirect domination or the unique outcome determined with certainty either in the RE approach or Kimya's coalitional behavior, is intended to extend the reasoning beyond the confidence assumption.

To motivate our solution concept, we consider a modification of Example 2 (Figure 3) in D&V (2017). Though formal definitions of the various farsighted solution concepts will be given only in Sections 2 and 4, the example is simple enough to explain the motivation for the introduction of the SPCS in view of some shortcomings of existing farsighted solution concepts.

Example 1.1 *Consider the four player abstract game associated with Figure 1.1 that has five states with possible transitions between states as illustrated in the figure. Specifically, the set of players is* $N = \{1, 2, 3, 4\}$*, the set of states is* $Z = \{\text{A,B,C,D,E}\}$*, the utility for each of the four players is displayed in the figure next to each state, and the effective coalitions that can move between states are depicted with arrows. Thus, for example coalition* $\{1\}$ *is the only coalition that can affect a move from State* A*, and such a move leads to State* C*. The aim is to identify the set of stable states, that is, the set of states from which farsighted*

players will not move. Clearly, both states D *and* E*, which are terminal states and movement therefrom is impossible, are stable. The identity of the player, either* 3 *or* 4*, that gets the opportunity to be the first to move from State* C *is not specified in an abstract game. Despite such uncertainty, farsighted reasoning naturally suggests that both Players* 1 *and* 2 *strictly prefer to move from their States* A *and* B*, respectively. Indeed, Player* 4 *strictly prefers to move from State* C *to States* E*, wherein Player* 4 *is strictly better off than at either State* C *or State* D*. Given the opportunity to move first at State* C*, Player* 3 *strictly prefers to move therefrom to State* D *in order to preempt such a move by Player* 4 *from State* C *to State* E*, wherein Player* 3 *would be strictly worse off than at* D*. Since both Players* 3 *and* 4 *strictly prefer to move from State* C*, the moves by Players* 1 *and* 2 *would yield for both of them some non-deterministic prospect of utilities* $1, 3$*, which both of them strictly prefer to the utility,* 1*, they have, for sure, at States* A *and* B*.*
According to the SPCS*, as will be further elaborated in Section 2, States* D,E *are reachable from State* C*, namely there is a subgame perfect equilibrium in which* D,E *are final states when the initial state is* C*. Consistency requires that all reachable and stable states are final states following a move from either* A *or* B*. This, in turn, together with subgame perfection, implies that Players* 1 *and* 2 *will move from States* A *and* B*, respectively. Thus, according to the* SPCS*, which extends farsighted stability beyond the confidence assumption, States* A,B *and* C *are not stable, and the* SPCS *in the above example consists only of the terminal states, i.e.,* {D,E}*.*
By contrast, in Section 4 we show that LCS = LCCS = vNM FSS = {B,D,E}*, whereas the RE solutions (including Kimya 2020) predict also the sets* {A,D,E} *and* {D,E}*. We conclude that only the SPCS prescribes/predicts optimal moves for Players* 1 *and* 2*. Farsighted players according to the SPCS would move from States* A *and* B *and realize a non-deterministic prospect of utilities* $1, 3$*, which is preferable to staying at States* A *and* B *according to which they will attain for certainty a utility of* 1*. As is clarified in Section 4, the RE based solutions fail because they subscribe to the confidence assumption. They rely on a deterministic expectation function (or the similar concept of coalitional behavior), which identifies, with certainty, the moving coalition, if any, at each state and the resulting new state.*

The SPCS shares with some existing expectation function farsighted solution concepts the use of vNM type of consistency. But differently, it does not identify a unique moving coalition, if any, at each state and the resulting new state. Rather, it relies on a strategy profile to ensure commonly held beliefs, endogenously determined, about the probabilities with which coalitions are selected to affect a change, if any, at each stage, and about the continuation paths following coalition moves. The SPCS could be history-dependent, and since the players are guided by subgame perfection, coalitions are employing best response moves, thus satisfy maximality. As in the case with other static farsighted solution concepts, the SPCS is a natural solution concept in settings in which actions are public, transient states have no payoff consequences, and payoffs are realized only when players reach a final agreement.

We investigate the SPCS under two different assumptions on the *endogenously* determined protocol, which, as part of the subgame perfect equilibrium, following any history generates the coalition that can affect a move from the current state. Under one assumption, the protocol always has full support, thus any of the coalitions that can affect a move from the current state are generated with positive probability by the equilibrium protocol, and maintain the notation of SPCS to denote the corresponding farsighted solution concept. Under the other assumption, called regular protocol, at least one coalition, if one exists, that can affect a move from the current state is generated with positive probability by the equilibrium protocol, and denote the corresponding solution concept as SPCS*. We prove existence of SPCS* for any finite game. In the sequel, unless explicitly stated otherwise, any reference to SPCS is implicitly also a reference to SPCS*.

We analyze and characterize our solution concept for the general case when coalition moves are possible. However, we are particularly interested to study situations in which only moves by individual players are allowed. This would correspond, e.g., to farsighted settings based on Cournot and Bertrand competition, where players are not allowed to collude or coordinate their actions. Indeed, we study the structure of the SPCS for the class of farsighted normal form games (possibly with a continuum set of states). In particular, we show that even when only single players can move, quite surprisingly, farsighted players who follow the reasoning of the SPCS (and SPCS*) would always be able to achieve (weak) Pareto efficiency in any normal form game having a pure Nash equilibrium. Thus, the SPCS reasoning leads to efficiency in "farsighted" versions of, e.g., Bertrand games, Cournot games, some instances of decentralized supply chains coordination problems, network formation, and the Prisoner's Dilemma. For all these settings, a SPCS is a singleton set consisting of a unique Pareto efficient outcome (up to equivalence to all players) weakly dominating a myopic (pure Nash) equilibrium outcome, which would be realized if the players are myopic. Such a Pareto efficient outcome is achieved using strategies similar to 'grim-trigger' strategies commonly used in folk theorems within the repeated games literature. An important contribution of our approach is that, contrary to many cooperative game models which assume Pareto efficiency from the start, we derive such efficiency within a noncooperative framework. This is undertaken using strategies that could be thought of as modelling tacit negotiation.

Our results on the SPCS (SPCS*), which was introduced to model farsighted behavior in abstract games and was surprisingly found to lead to Pareto efficiency in normal form games having a Nash equilibrium, are indeed somewhat related to the vast literature on repeated games, wherein conditions are provided that possibly induce cooperation in non-cooperative settings. That is, like in repeated games, where playing repeatedly the same non-cooperative game may lead to cooperation and efficiency, our results reveal that players who follow the logic of the SPCS would cooperate in farsighted versions of normal form games having a pure Nash equilibrium. However, we note that while 'grim-trigger' strategies in repeated games may lead to cooperation, there are also equilibria in which the players may fail to do so. For example, in a repeated games setting of the Prisoners' Dilemma, repeated defection at each stage may also be an equilibrium behavior by the two players. By contrast, as mentioned above, in the farsighted version of the Prisoners' Dilemma with individual moves, the SPCS consists uniquely of the strategy profile corresponding to the socially optimal outcome. Consequently, contrary to repeated games in which efficiency is achieved as a possibility, the SPCS achieves efficiency as a necessary implication.

We further show that the SPCS reasoning leads to efficiency also for normal form games having no myopic equilibrium, given that one is willing to accept as possible an undesirable outcome of the game corresponding to a 'swinging' behavior in which some of the players keep on moving between states. We show that all the above results continue to hold when coalition moves are permitted, where in this case a myopic equilibrium is a state from which no coalition strictly prefers a single move.

In some sense, our approach for farsightedness in abstract games is related to that of Herings et al. (2004), who associate with a finite abstract game a non-cooperative finite horizon multistage game with observed actions, and apply extensive-form rationalizability in the sense of Pearce (1984) on this game in order to define their solution concept, consisting of the socially realizable outcomes. They show that their solution concept is not empty and that it satisfies a certain coalitional rationality property.

In the above approaches for farsightedness the players only care about the final outcome the negotiations lead to, and are referred to in the literature as static. In the dynamic approach to farsightedness, see, e.g., Konishi and Ray (2003) and Ray and Vohra (2015a), with their equilibrium process of coalition formation (EPCF) concept of solution, players get discounted state-dependent payoffs at each state. In Kimya's (2020) study of extended coalition games, the utilities of players are defined over the paths of play, which allows his model to accommodate both the static and dynamic approaches.

Finally, we note that farsighted solution concepts and various refinements and modifications thereof, were applied in a variety of settings. For example, they were employed in studies on coalition formation (e.g., M&V 2004, Granot and Yin 2008, Nagarajan and Sošic

2007, Herings et al. 2010), efficiency in some classes of normal form games[3] (e.g., Suzuki and Muto 2005, Kawasaki 2015, and Bloch and van den Nouweland 2021), network stability (e.g., Page et al. 2005, Dutta et al. 2005, Herings et al. 2009, Page and Wooders 2009, Kimya 2020, and Luo et al. 2021), patent licensing negotiation (Hirai et al. 2019), hedonic games (Diamantoudi and Xue 2003, 2007), matching (Mauleon et al. 2011) and formation of alliances (Cai and Kimya 2023). Not surprisingly, farsighted reasoning could yield quite different insights than those derived from myopic considerations. For example, in coalition formation studies (e.g., M&V 2004, and Granot and Yin 2008), it is shown that under certain conditions, farsighted players would adhere to the grand coalition to the benefit of all players. By contrast, myopic players would defect and form the stand-alone coalition structure, to the detriment of all players.

The plan of this paper is as follows. In Section 2 we introduce the farsighted game, formally define the SPCS and the SPCS* farsighted solution concepts, investigate some of their basic properties, and prove the existence of the SPCS* for any finite game. In Section 3 we introduce farsighted normal form games, derived from standard one-shot normal form games by allowing players, or coalitions, to publicly and repeatedly, in their turn, change their previous actions. For this class of games, we prove that any set of outcomes weakly dominating a myopic equilibrium is a SPCS (and SPCS*) if, and only if, it consists of a single Pareto efficient outcome (up to equivalence to all players). The analysis is extended to farsighted normal form games without a myopic equilibrium. We illustrate in Section 3 the significance of our findings by demonstrating, for example, that by contrast with existing farsighted solutions such as the LCS and the LCCS, but similarly, e.g., to the vNM FSS, farsighted players who follow the SPCS reasoning are always able to share the monopolistic profit in farsighted settings based on Bertrand and Cournot competition. We further show that farsighted players who follow the SPCS reasoning are always able to achieve full coordination and Pareto efficiency in decentralized supply chain contracting even when they cannot form coalitions. Finally, we model a network formation process as a farsighted normal form game involving only individual players, and use the SPCS to generalize existing results which mollify the tension between stability and efficiency in network formation. In Section 4 we briefly survey related farsighted solution concepts in the literature and compare them, including examples, to the SPCS. Section 5 provides concluding remarks. All proofs are collected in an appendix.

[3]See further Section 3.1.

2 Model and Solution: The Subgame Perfect Consistent Set

Consider a dynamic game in which players can act repeatedly and publicly by moving between states they care about. For example, one may think of a group of firms engaged in contract negotiation, in which a state is a vector of contract parameters set by the firms, and each firm is responsible for a different part of the vector. Players only care about the final state reached, either in an actual negotiation (e.g., Hirai et al., 2019) or in a tacit negotiation (e.g., Suzuki and Muto, 2006), irrespective of the sequence of actions that lead to it. Thus actions have no significant costs, and the time frame in which they are taken is short with no relevant discounting of the final state utility. Despite the dynamic nature of the game, it can be thought of as a one period interaction between the players.

In this section we formally study such a dynamic game for the purpose of introducing and analyzing our farsighted solution concept, the SPCS. We first present in Section 2.1 our Farsighted Game model, and subsequently, subgame perfect equilibrium in Section 2.2, the SPCS approach in Section 2.3, and some initial analysis in Section 2.4 via an example and several general results, including existence.

2.1 Farsighted Game

The setting can be described as an abstract game (Chwe 1994, see also Greenberg 1990), denoted by $\langle N, Z, (u_i)_{i\in N}, (\rightarrow_S)_{S\subseteq N}\rangle$, where: N is a non-empty, finite set of players; Z is a non-empty, measurable set of states; the utility function $u_i : Z \rightarrow \mathbb{R}$ for each $i \in N$ determines player i's utility from a state $z \in Z$ (when it is a final state); and the binary relation $\rightarrow_S \subseteq Z \times Z$ for each $S \subseteq N$ describes the players ability to change the current state, where $z^1 \rightarrow_S z^2$ for $z^1, z^2 \in Z$ means that coalition S of players can alter the current state z^1 by moving to a new state z^2 (with the empty coalition always having null effectiveness, i.e. $\rightarrow_\emptyset = \emptyset$). A state $z \in Z$ is said to be terminal if there is no $z' \in Z$, $z' \neq z$, and $S \subseteq N$, such that $z \rightarrow_S z'$. For example, in the contract negotiation setting, Z can be the set of all contract parameter vectors possibly set by the firms, and the effectiveness of a singleton coalition $S = \{i\}$, consisting of a single player i, allows the player to alter the current state by changing their own contract part without changing the parameters set by the other players. Furthermore, in this example, the effectiveness of a coalition consisting of two or more players can describe agreements between members of the coalition that allow particular simultaneous changes in the parameters set by all members of the coalition (again without changing the parameters set by players outside the coalition).

As mentioned above, we envision an extensive form game with perfect information, in which all actions are public, which we call the Farsighted Game (FG). The complete description of the game is given by the following sequence of events. First, an initial state in Z is publicly chosen by a dummy player c, referred to as nature, who is assumed to be indifferent between all states in Z. Then, an infinite sequence of stages initiates, each stage consisting of the following two steps: (Step 1) some coalition is publicly selected by nature; and (Step 2) the selected coalition can publicly keep the current state unchanged or move to a new state in Z according to its effectiveness. The assumption that nature selects the initial state is taken only to reflect the fact that none of the players in N can affect this choice – one could alternatively define the game with a prespecified initial state instead of selection by nature. Note that in a setting where players can repeatedly observe the actions made by others and adjust their own actions accordingly, full farsightedness may necessitate unbounded play: actions are made taking into account the fact that any move by a coalition may be counteracted with a further move by another coalition, without limit. Such an approach is in the spirit of modelling farsighted negotiation. It is natural therefore that players only care about the final outcome of the negotiation process, with no intermediate payoffs.

Formally, for the extended set of players $N_c = N \cup \{c\}$, define the set $\mathcal{H}$ of all possible histories in the game as the set of all possibly infinite sequences $h = (h_k)_{k=0}^{K_h}$, including the empty sequence when $K_h = -1$, such that h_0 is some initial state in Z, for all odd numbers $k \geq 1$, h_k is some coalition $S \subseteq N$, and for all even numbers $k \geq 2$, $h_k \in Z$, such that $h_k = h_{k-2}$ or $h_{k-2} \rightarrow_{h_{k-1}} h_k$. For two finite histories $h, h' \in \mathcal{H}$ with even, positive cardinality such that $h'_0 = h_{K_h-1}$ or $h_{K_h-1} \rightarrow_{h_{K_h}} h'_0$, denote by (h, h') the history in $\mathcal{H}$ obtained when h is followed by h'. Denote the set of infinite histories by $\mathcal{H}_\infty$. Define the player function $P : \mathcal{H} \setminus \mathcal{H}_\infty \rightarrow N_c$ by $P(\emptyset) = c$, $P(h) = c$ for every finite history h with odd cardinality $|h|$, and $P(h) = h_{K_h} \subseteq N$ for every finite history h with even, positive cardinality.

We say that a history $h \in \mathcal{H}$ converges for player i if there exist $\bar{z}_i(h) \in Z$ and an even, positive number $k_{h,i}$ such that $u_i(h_k) = u_i[\bar{z}_i(h)]$ for every even k such that $k_{h,i} \leq k \leq K_h$. Let $k^0_{h,i}$ to be the minimal such $k_{h,i}$. A history h converges when it converges for all players, in which case we omit the player index and just write $\bar{z}(h)$ and k^0_h. Convergence occurs, in particular, when $h_{k^0_h}$ is a terminal state, or when h is finite.[4] Denote by $\bar{\mathcal{H}}$ (resp., $\bar{\mathcal{H}}_i$) the set of all infinite converging histories (resp., for player i). When an infinite history h does not converge for player i, we say that h leads for that player to 'swinging', denoted by w, and define $\bar{z}_i(h) = w$ and $k^0_{h,i} = \infty$. Let $\bar{Z} = Z \cup \{w\}$. Extend the utility function u_i for each player i to $\bar{Z}$ by defining $u_i(w) = -\infty$ for all i (see e.g., Harsanyi 1974, Mariotti 1997 and

[4]Our results would continue to hold if the stronger requirement $h_k = \bar{z}(h)$ was used instead of $u_i(h_k) = u_i[\bar{z}_i(h)]$ to define convergence.

Flesch et al. 2010, Kimya 2020, K&R 2021)[5]. This assumption fits well with a negotiation setting, as swinging behavior is similar to a disagreement outcome which all players prefer to avoid. Nevertheless, we emphasize that swinging exists in our model in order to provide a complete description of what might happen following any history. Importantly, our solution concept does not rely on threats with swinging, on or off equilibrium path, in order to support some outcome.

We would like our solution concept not to assume an exogenously given, specific process which determines the coalition selected to make a choice. Therefore, in the FG, nature's behavior strategy, named protocol, is determined endogenously in the solution together with the strategies of all coalitions. The protocol is a function σ_c, defined over $\{h \in \mathcal{H} \setminus \mathcal{H}_\infty \mid P(h) = c\}$ such that $\sigma_c(h)$ at history h for which $P(h) = c$ is a probability measure defined over Z when h is the empty sequence and over 2^N otherwise, specifying the distribution over initial states and over the coalitions selected to make a choice. Say that protocol σ_c is regular if for every finite history h with odd cardinality $|h|$, $\sigma_c(h)(S) > 0$ for some coalition S, if one exists, that has the effectiveness to move at the current state, i.e. S with $\{z \in Z \mid z \neq h_{K_h-1}, h_{K_h-1} \rightarrow_S z\} \neq \emptyset$. Say that protocol σ_c is full support if $\sigma_c(h)$ has full support at every history h for which $P(h) = c$. For coalitions, randomizations are considered in the analysis but are never implemented. A pure behavior strategy of a coalition S is a function $\sigma_S : \{h \in \mathcal{H} \setminus \mathcal{H}_\infty \mid P(h) = S\} \rightarrow Z$ such that $\sigma_S(h) \in \{z \in Z \mid z = h_{K_h-1}$ or $h_{K_h-1} \rightarrow_S z\}$, specifying the action taken by this coalition after any history in the game where this coalition is selected to make a choice. A strategy profile is a vector $\sigma = [\sigma_c, (\sigma_S)_{S\subseteq N}]$ specifying the protocol σ_c and the strategy σ_S of each coalition S. Denote by σ_{-S} the protocol σ_c together with the vector of strategies for all coalitions except for S.

2.2 Subgame perfect equilibrium

We consider subgame perfect equilibria of the FG. For the purpose of defining such equilibria, it is sufficient to consider subgames beginning with coalitions' decisions, i.e. following finite histories h with even, positive cardinality. Following such h, a strategy profile σ generates a probability measure over the infinite histories h' beginning with h that are dictated by σ. Let Σ_h denote the set of strategy profiles which, following h, generate a probability measure having support consisting only of infinite converging histories. Our notion of farsightedness relies on such strategy profiles for all h. A strategy profile $\sigma \in \Sigma_h$ generates for player i a

[5] The results in this paper would remain unchanged if instead one adopted the approach of positive recursive games (e.g., Flesch et al. 2010), by normalizing u_i to be non-negative with $u_i(w) = 0$, associate any infinite history h with an infinite sequence of payoffs consisting of a finite prefix of zero payoffs from transient/non-absorbing states followed by an infinite, non-negative and constant suffix of the convergent/absorbing state payoff $u_i[\bar{z}_i(h)]$, and evaluate this sequence using the limit of the means criterion.

distribution $d_{\sigma|h}$ over $\bar{z}(h') \in Z$. This distribution has countable support because there are finitely many coalitions that may be selected and convergence occurs after a finite history. Elements of this support are called final states. To define $d_{\sigma|h}$, let

$$\Phi_{\sigma|h} = \left\{ (h'_k)_{k=0}^{k^0_{h'}} \;\middle|\; \begin{array}{c} h' \in \bar{\mathcal{H}}, h'_k = h_k, \forall k \in 0, 1, \ldots, K_h \text{ and} \\ h'_{k+1} = \sigma_{h'_k}[(h'_l)_{l=0}^k], \forall k = K_h, K_h + 2, \ldots \end{array} \right\}.$$

The set of final states according to strategy profile σ following history h, i.e. the support of $d_{\sigma|h}$, is

$$\Psi_{\sigma|h} = \{\bar{z}(h') \mid h' \in \Phi_{\sigma|h}\},$$

and then $d_{\sigma|h}(z) = \sum_{\{h' \in \Phi_{\sigma|h} | \bar{z}(h') = z\}} \prod_{k=K_h+1, K_h+3, \ldots, K_{h'}-1} \sigma_c((h'_l)_{l=0}^{k-1})(h'_k)$ for $z \in Z$. Given $\sigma \in \Sigma_h$, a deviation of coalition S away from σ_S may lead to a strategy profile not in Σ_h. Therefore, more generally, any strategy profile σ generates for player i a distribution $d_{i,\sigma|h}$ over $\bar{z}_i(h') \in \bar{Z}$, still with countable support because there are finitely many coalitions that may be selected and convergence to any $z \neq w$ occurs after a finite history. Similarly, for each player i define $\Phi_{i,\sigma|h}$, $\Psi_{i,\sigma|h}$ and $d_{i,\sigma|h}$ relying on $\bar{\mathcal{H}}_i, \bar{z}_i(h')$ instead of $\bar{\mathcal{H}}, \bar{z}(h')$, and let $d_{i,\sigma|h}(w) = 1 - \sum_{z \in \Psi_{i,\sigma|h} \cap Z} d_{i,\sigma|h}(z)$.

Subgame perfection in our approach is required with respect to coalitional risk preference extensions obeying first order stochastic dominance and coalitional dominance consistent with the given player utilities u_i. In a subgame that follows a finite history h with $P(h) = S$, coalition S has a strict preference relation $\succ_{S,h}$ over strategy profiles, where weak preference $\succsim_{S,h}$ and indifference $\sim_{S,h}$ are defined from the strict preference $\succ_{S,h}$ in the usual way. When the coalition is a singleton player i, we assume the following.

Assumption 2.1 *For each player i, the preference relation $\succ_{i,h}$ is continuous, and $\sigma \succ_{i,h} \sigma'$ is implied if $d_{i,\sigma|h}$ strictly first order stochastically dominates $d_{i,\sigma'|h}$, where $\bar{Z}$ is ordered according to u_i.*

Assumption 2.1 allows in particular for expected utility preferences. For our analysis, the important implication of this assumption arises when considering a strategy profile which leads to a degenerate distribution over final states, i.e. some $z^0 \in Z$ for sure (with probability 1). Such a strategy profile is strictly worse than any strategy profile leading to a non-degenerate distribution over final states which assigns positive probability only to states in Z at least as good as z^0.

For general coalitions we assume the following.

Assumption 2.2 *For each coalition S, the preference relation $\succ_{S,h}$ satisfies that $\sigma \succ_{S,h} \sigma'$ is implied if $\sigma \succsim_{i,h} \sigma'$ for all members $i \in S$, with strict preference for at least one member of S.*

Given coalition preferences, we can define a strategy profile σ as a subgame perfect equilibrium if $\sigma \succsim_{S,h} (\hat{\sigma}_S, \sigma_{-S})$ for every coalition S, every finite history h with $P(h) = S$, and every strategy $\hat{\sigma}_S$ for this coalition.

2.3 SPCS

In analyzing the farsighted game described above, we seek a solution concept that produces a set of states considered farsighted stable. As mentioned in the Introduction, our proposed solution is based on subgame perfection. In our approach, subgame perfection replaces the various prescriptions embodied in all other farsighted solution concepts as to where a move by a coalition, S, may lead to. To formalize where a move might potentially lead to under subgame perfection, we say that a state z^2 is reachable from a state z^1 if there exists a subgame perfect equilibrium strategy profile σ, with $\sigma \in \Sigma_h$ following any finite history h with even, positive cardinality, such that z^2 is a final state (i.e., the play converges to it with positive probability) in a subgame in which z^1 is the initial state. Denote by $R(z^1)$ the set of states z^2 reachable from z^1 with full support protocol σ_c, and by $R^*(z^1)$ the set of states z^2 surely (i.e., with probability 1) reachable from z^1 with regular protocol σ_c. For a state z to be a final state according to a subgame perfect equilibrium, it must be reachable from itself.

We note that a solution concept defined as the vNM stable set with respect to the reachability relation R or R^* would not exist in many simple examples, e.g., in the roommate game. By contrast, we show (Theorem 2.1), for example, that our solution concept exists for all finite games. Furthermore, by contrast with, e.g., the RE approach, coalition S does not know for certain where its move would end up, since it does not know with certainty the identity of the coalition that will move at each state and the state it would move to, until a final stable state is reached, if at all. Accounting for this inherent uncertainty requires departure from the standard defition of vNM stability.

Essentially as all other farsighted solution concepts, such as those based on the RE approach, as well as the LCS and the vNM FSS, our solution concept, the SPCS, is based on a consistency notion in the spirit of vNM stable sets (von Neumann and Morgenstern 1944). Roughly speaking, this consistency notion requires the solution set X of states to satisfy the following two properties: (i) farsighted internal consistency: for each state $z \in X$, no coalition prefers to move away from z, anticipating that such a move would eventually end up in X; and (ii) farsighted external consistency: for each state $z \notin X$, there always exists a coalition that prefers to move away from z, again anticipating that such a move would eventually end up in X. This leads to the following definition.

Definition 2.1 *A set of states $X \subseteq Z$ is a* Subgame Perfect Consistent Set *(*SPCS*) if there exists a subgame perfect equilibrium σ with full support protocol σ_c such that the following three requirements are satisfied:*
(a) for any history $h = (z, S^1, z, S^2, \ldots, z, S^t)$ such that $z \in R(z)$ and $\{S^l\}_{l=1}^{t-1} \supseteq 2^N \setminus \emptyset$, $\sigma_{S^t}(h) = z$; and
(b) for any history $h = (z^1, S^1, z^1, S^2, \ldots, z^1, S^t, z^2, S^{t+1})$ such that $z^1 \neq z^2$ and $\{S^l\}_{l=1}^{t-1} \not\supseteq 2^N \setminus \emptyset$, the set of final states is $\Psi_{\sigma|h} = X \cap R(z^2)$; and
(c) $z \in X$ if, and only if, $\sigma_S(h) = z$ for any coalition S and any finite history $h = (z, S^1, z, S^2, \ldots, z, S)$.
A set of states $X \subseteq Z$ is a SPCS* *if the protocol σ_c is only required to be regular, and with R replaced with R^*.*

Requirement (a) says that after choices by all coalitions to stay at an initial state $z \in Z$ reachable from itself, z is the final state for sure. According to requirement (b), after an initial move from an initial state by some coalition, all reachable states in X from z^2, and only them, are final states. Both requirements hold on or off equilibrium path. Requirement (c) incorporates the farsighted internal and external consistency properties described before Definition 2.1, which therefore form an additional sense of fixed point apart from the usual one delivered by equilibrium: The set X is exactly the set of states from which, on equilibrium path, no coalition moves when selected as initial states. Whenever X satisfies Definition 2.1 with respect to σ, we say that σ supports X as a SPCS (or SPCS*).

2.4 Initial analysis

We first demonstrate our solution by considering the example presented in the Introduction.

Example 2.1 *We apply the* SPCS *(resp.,* SPCS**) to the game associated with Figure 1.1. A first step in the analysis is to establish the reachability functions R, R^*. To this end, consider the strategy profile $\bar{\sigma}$ defined for all $h \in \mathcal{H} \setminus \mathcal{H}_\infty$ as follows: (i) $\bar{\sigma}_c(h)$ for $P(h) = c$ assigns equal probabilities to each $S \subseteq N$, and (ii) $\bar{\sigma}_S(h)$ for $P(h) = S$ is equal to* C *if $S = \{1\}$ and h ends with $k_{K_h-1} =$* A *or if $S = \{2\}$ and $k_{K_h-1} =$* B*, it is equal to* D *if $S = \{3\}$ and $k_{K_h-1} =$* C*, it is equal to* E *if $S = \{4\}$ and $k_{K_h-1} =$* C *, and it is equal to k_{K_h-1} otherwise. According to $\bar{\sigma}$, each coalition is equally likely to be selected to make a choice whether to keep the current state or to move to a new state, and all coalitions move to the* *unique possible state whenever they can. The set of final states following any history h with $P(h) \subseteq N$ is $\Psi_{\bar{\sigma}|h} = \{\text{D,E}\}$ if $k_{K_h-1} \in \{\text{A,B,C}\}$ and $\Psi_{\bar{\sigma}|h} = \{k_{K_h-1}\}$ when $k_{K_h-1} \in \{\text{D,E}\}$. Furthermore, following any history h with $P(h) = S$, if $S = \{3\}$ and $k_{K_h-1} = \text{C}$ then $\bar{\sigma}$ leads to utiliy 2 to Player 3 following their move from State C to State D, which is strictly better than the uniform distribution over $\{2, 0\}$ if this player stays and then with equal probabilities either $\{3\}$ is selected and accordingly moves to D or $\{4\}$ is selected and accordingly moves to E. Similarly, if $S = \{4\}$ and $k_{K_h-1} = \text{C}$ then $\bar{\sigma}$ leads to utility 3 to Player 4 following their move from State C to State E, which is strictly better than the uniform distribution over $\{0, 3\}$ if this player stays and then with equal probabilities either $\{3\}$ moves to D or $\{4\}$ moves to E. If $S = \{1\}$ and $k_{K_h-1} = \text{A}$ or $S = \{2\}$ and $k_{K_h-1} = \text{B}$ then $\bar{\sigma}$ leads to some non-degenerate distribution over $\{3, 1\}$, which is strictly better than the sure utility of 1 if this player stays. Therefore $\bar{\sigma}_S \succsim_{S,h} \sigma'_S$ for any strategy σ'_S and $\bar{\sigma}$ is a subgame perfect equilibrium. Since any other subgame perfect equilibrium does not change $\Psi_{\bar{\sigma}|h}$ following any history h, the reachability functions R, R^* coincide and are given by $R(\text{A}) = R(\text{B}) = R(\text{C}) = \{\text{D,E}\}$, $R(\text{D}) = \{\text{D}\}$ and $R(\text{E}) = \{\text{E}\}$.*

Given R, R^*, $X = \{D, E\}$ is a SPCS (resp., SPCS*) for this example, supported by the strategy profile $\sigma^X = \bar{\sigma}$. To see this, note that requirement (a) of Definition 2.1 is satisfied since State D and State E are all the states satisfying $z \in R(z)$ (resp., $z \in R^*(z)$), and both are terminal states. Requirement (b) is satisfied since $k_{K_h-1} \in \{\text{A,B,C}\}$ implies $\Psi_{\bar{\sigma}|h} = \{\text{D,E}\} = X \cap R(k_{K_h-1})$ and $k_{K_h-1} \in \{\text{D,E}\}$ implies $\Psi_{\bar{\sigma}|h} = \{k_{K_h-1}\} = X \cap R(k_{K_h-1})$. Requirement (c) is satisfied since $k_{K_h-1} \in \{\text{A,B,C}\}$ implies $\sigma_S(h) \neq k_{K_h-1}$ for, resp., $S = \{1\}, \{2\}$, and $\{3\}$ or $\{4\}$, and $k_{K_h-1} \in \{\text{D,E}\}$ implies $\sigma_S(h) = k_{K_h-1}$ for all S. Since any other subgame perfect equilibrium does not change $\Psi_{\bar{\sigma}|h}$ following any history h, there is no other SPCS (resp., SPCS*).

We now investigate some basic properties of our solution concept, starting with non-emptiness.

Proposition 2.1 *If a SPCS or SPCS* exists, then it is non-empty, and includes all terminal states.*

The following example demonstrates that existence is not guaranteed. In this example the failure is due to the lack of a subgame perfect equilibrium.

Example 2.2 *There is one player, Z is the set of natural numbers, $u(z) = z$ for each $z \in Z$, and the player can only move from $z = 1$ to any $z > 1$ (with no other possible moves included in the effectiveness relation). In this example there is neither a SPCS or SPCS* because there is no subgame perfect equilibrium: when $z = 1$ is selected as an initial state, for any choice the player can make there exists a strictly better choice.*

Despite such examples, the following proposition provides a sufficient condition for existence.

Theorem 2.1 *Whenever Z is finite, there exists a* SPCS^*.

The argument in the proof of Theorem 2.1 initially shows that some state is surely reachable from any state; this is then used in the construction of a set X using a finitely terminating iterative procedure, for which a corresponding strategy profile σ is explicitly constructed, which is then shown to support X is a SPCS^*.

When analyzing farsighted normal form games in Section 3, we will show that the assumption that Z is finite is not necessary for existence. Indeed, a SPCS, as well as a SPCS^*, can exist also for games with an infinite Z.

3 Farsighted Normal Form Games

In this section we analyze normal form games when they are viewed as farsighted games.

Definition 3.1 *A k-normal form game is an abstract game $\langle N, Z, (u_i)_{i\in N}, (\to_S)_{S\subseteq N}\rangle$ such that k is integer with $1 \leq k \leq |N|$, $Z = \times_{i\in N} A_i$, where A_i is the set of alternatives available to player i, and for each $z^1, z^2 \in Z$ and each coalition $S \subseteq N$, $S \neq \emptyset$, the effectiveness relation $z^1 \to_S z^2$ holds if, and only if, $|S| \leq k$ and $z^1_i = z^2_i$ for each $i \notin S$. A normal form game is a k-normal form game for some k.*

In view of Definition 3.1, our analysis of normal form games allows for the typical restriction to only individual player moves when $k = 1$, but also allows for coalitional moves whenever $k > 1$. We view the concept of a pure Nash equilibrium as myopic, as it does not involve farsighted reasoning. Formally, a myopic equilibrium is defined as follows.

Definition 3.2 *In a normal form game, $e \in Z$ is a myopic equilibrium if $u_i(e) \geq u_i(z)$ for each $i \in S \subseteq N$ and $z \in Z$ such that $e \to_S z$.*[6]

We show that the SPCS solution approach provides a surprising and striking conclusion in the farsighted analysis of normal form games: whenever the game possesses a myopic equilibrium, Pareto efficiency is always and necessarily achieved. Moreover, Pareto efficiency is achieved even when coalition moves are not permitted.

[6] Our analysis also applies when considering Stackelberg games (and extensive form games with perfect information in general) in their reduced normal form. This is justified by the view that in the farsighted perspective, the leader and the follower are indistinguishable. Thus, in this case, the results concerning farsighted normal form games apply when replacing pure Nash equilibria with pure Stackelberg equilibria.

When analyzing farsighted normal form games we assume that each player's utility function is continuous, and that a SPCS / SPCS* is required to be a compact set. To state the main result we need some additional pieces of terminology/notation. We say that a state $z \in Z$ is Pareto efficient if there does not exist a state $z' \in Z$ such that $u_i(z') > u_i(z)$ for all $i \in N$. Further, we write that $Y \approx \{z^*\}$ whenever a subset $Y \subseteq Z$ is countable[7] and consists of states that are all equivalent for all players, i.e. there exists $z^* \in Y$ such that $u_i(z') = u_i(z^*)$ for each player i and $z' \in Y$. We can now state our main result.

Theorem 3.1 *(1) For any normal form game, if $X \subseteq Z$ is a SPCS or SPCS*, then $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$; and*
(2) For any normal form game, if $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$ such that $u_i(z^*) \geq u_i(e)$ for some myopic equilibrium $e \in Z$ and all $i \in N$, then X is a SPCS and SPCS.*

Intuitively, the uniqueness of a SPCS or SPCS*, up to equivalence, stems from the sure reachability relation. Namely, we prove, in Theorem 3.1, that every state in a SPCS, X, is surely reachable from any state in Z. If not all states in X are equivalent for all players, there is a strictly worse state z in X for some player i. Then, since a move by player i from state z must end up at X (Definition 2.1(b)), and any state in X is surely reachable after such a move, player i would prefer to move from state z (see Assumption 2.1). Indeed, she cannot be worse off by such a move, and could possibly be strictly better off from it. Thus, state z cannot be stable and does not belong to X. The Pareto optimality of the unique (up to equivalence) z^* in X is shown to follow from the reachability from itself of any state d which Pareto strictly dominates x^*, if such a state d exists. Indeed, intuitively, in a subgame starting at state d, coalitions will not to move therefrom since any such move, by Definition 2.1(b), must end up at x^* at which they are all strictly worse off. This intuition is shown to imply, by Definition 2.1(a,c), that state d is in X, contradicting the uniqueness (up to equivalence) of z^* in X.

For a 1-normal form game, as mentioned in the Introduction, the strategies supporting a SPCS or SPCS* in Theorem 3.1 are similar to 'grim-trigger' strategies commonly used in folk theorems within the repeated games literature (see, e.g., Osborne and Rubinstein 1994), adapted to our setting where a repeated stage game is not required. The strategies involve a threat of reaching an undesirable outcome off equilibrium path in order to create incentives to reach a good outcome on equilibrium path. These strategies could be thought of as social norms that are publicly known to all players in the game. Since no player has an incentive to

[7] As shown in the proof of Theorem 3.1, whenever $X \subseteq R(z^1)$ for any $z^1 \in Z$, requirement (b) of Definition 2.1 implies that X is necessarily a countable set.

diviate from acting according to these social norms, they are indeed implemented, leading to the outcome of the corresponding SPCS / SPCS*. But note that contrary to folk theorems in repeated games, which do not claim to achieve efficiency, we achieve efficiency as a necessary implication. Thus the contribution of Theorem 3.1 is the conclusion that whenever the normal form game possesses a pure Nash equilibrium, such social norms must lead to Pareto efficiency, which is achieved rather than assumed.

Theorem 3.1 implies that a game with no Pareto efficient states cannot have a SPCS or SPCS*. Existence of Pareto efficient states is a mild condition satisfied in many interesting settings as demonstrated in the following Example.

Example 3.1 *Farsighted Prisoner's Dilemma. In this* 1*-normal form game there are two players, player* 1 *(the row player) and player* 2 *(the column player), each having two available alternatives and utilities as in the following matrix.*

	D	C
D	$1,1$	$4,0$
C	$0,4$	$3,3$

For this game there is a unique SPCS $X = \{(\text{C,C})\}$*. This SPCS is supported by the following strategy profile* σ^X*: after a selection of any initial state, each player moves to the alternative* C *and then does not move anymore, unless some player previously did not do so (or, both players did not move from* (D,D)*), in which case each player moves to the alternative* D *and then does not move anymore (see also proof of Theorem 3.1). As a result, the SPCS consists of the cooperative outcome,* (C,C)*, in the game. Note that for this example we show in Section 4 that the LCS, the LCCS, the vNM FSS and any rational expectation function based farsighted solution concept (all these solution concepts are described in Section 4), when it exists for this example, is equal to* $\{(\text{C,C}), (\text{D,D})\}$*.*

Herings et al. (2004) have shown that their solution concept satisfies coalitional rationality. That is, they have considered the social environment with a set of players N, set of outcomes, $Z = \{x_0, x_1, \ldots, x_k\}$, only the moves, $x_0 \to_N x_j$, $j = 1, \ldots, k$, are possible, and one outcome strictly dominates all other outcomes. Specifically, for all $i \in N$ and $j \neq 0, k$, $u_i(x_k) > u_i(x_j) > u_i(x_0) = 0$. They have shown that in this social environment, the Pareto-dominating outcome, x_k, is selected by each coalition. Each individual only agrees to move to the Pareto dominating outcome, and blocks all other moves. We can naturally cast the above social environment, used by Herings et al. (2004), as a farsighted normal form game, in which the set of strategies of each player is $\{x_0, x_1, \ldots, x_k\}$, and for each $j = 1, \ldots, k$, outcome x_j would be realized if all players chose to play strategy x_j, and otherwise, each player i would realize a utility $u_i(x_0)$. Furthermore, as assumed by Herings et al. (2004), for $i \in N$, and $j \neq 0, k$, $u_i(x_k) > u_i(x_j) > u_i(x_0) = 0$. Then the SPCS in this game is $\{(x_k, \ldots, x_k)\}$, since it is a Nash equilibrium in the associated normal form game which strictly Pareto dominates all other strategy profiles. Thus, similar to Herings et al. (2004) solution concept, the SPCS can also be viewed as satisfying coalitional rationality in the same social environment considered therein even without resorting to coalitions.

Theorem 3.1 provides sufficient but not necessary conditions for a Pareto efficient state to form a SPCS. Indeed, as demonstrated in the following example, we may have normal form games where $X = \{z^*\}$ is a SPCS for some Pareto efficient z^* that does not weakly dominate any myopic equilibrium.

Example 3.2 *Consider a farsighted* 1*-normal form game with two players,* 1 *and* 2*, corresponding to the row and column players, respectively, where each player has two alternatives and the utilities are described by the following matrix.*

	L	R
T	$1,1$	$1,2$
B	$0,2$	$2,2$

In this game the unique myopic equilibrium is (B,R)*, and thus is reachable from itself. By Theorem 3.1, the set* $X = \{(\text{B,R})\}$ *is a* SPCS*. Note that* (B,L) *is not reachable from itself, as player* 1 *can move and ensure the higher payoff* 1*. Similarly,* (T,L) *is not reachable from itself, as player 2 can move and ensure the higher payoff* 2*. But note that* (T,R) *is reachable from itself, and moreover, the set* $X = \{(\text{T,R})\}$ *is also a* SPCS*. This is true due to the following strategy profile: at* (T,R) *both players stay; at* (B,R) *player* 1 *stays and player* 2 *moves to* (B,L)*, unless* 2 *previously stayed, in which case they both stay and* (B,R) *becomes the final state; at* (B,L) *player* 1 *moves to* (T,L) *and player* 2 *stays; at* (T,L) *player* 1 *stays and player* 2 *moves to* (T,R)*. To see that indeed this strategy profile forms a subgame perfect equilibrium note that an attempt of* 1 *to improve by moving to* (B,R) *fails because* 2 *punishes by moving to* (B,L)*;* 2 *does not mind moving to* (B,L) *because* 1 *would move to* (T,L)*, in which case* 2 *would move and end up at* (T,R)*; if, on the other hand,* 2 *stays at* (B,R) *then* 1 *stays there also, leading to payoff* 2 *for player* 2*, which is not strictly better than the payoff at* (T,R)*.*

The following example demonstrates the possibility of existence of a SPCS in a farsighted normal form game despite the non-existence of a myopic equilibrium.

Example 3.3 *Farsighted Matching Pennies. Consider a* 1*-normal form game with two players (R and C), where each player has two alternatives and the utilities are described by the following matrix.*

$1, -1$	$-1, 1$
$-1, 1$	$1, -1$

In this game there is no myopic equilibrium, so Theorem 3.1 does not apply. Still, any set consisting of a single state in Z is a SPCS, *supported by a strategy profile similar to the one used in Example 3.2, i.e., the two players always stay at the single state in the given* SPCS, *and at any other state the player that receives utility -1 moves, unless both previously stayed there, in which case no one moves anymore. In this case, the player that receives utility -1 in the single state of the given* SPCS *prefers to accept this utility than to insist on moving, which would lead to swinging with utility $-\infty$. In this way, every state is reachable from itself. Each diagonal in the matrix is also a* SPCS, *supported by a strategy profile in which the players move only from states not on the diagonal.*

We now provide an extension of the analysis above that applies to normal form games having no myopic equilibrium. A game is said to be generic if $u_i(z) \neq u_i(z')$ for any player i and any two distinct states z, z'. Furthermore, in the definition of reachability, consider omitting the requirement that $\sigma \in \Sigma_h$ following any finite history h with even, positive cardinality. Under this extended reachability and the genericity assumption, Theorem 3.1 can be strengthened to an if and only if statement: for any generic normal form game, $X \subseteq Z$ is a SPCS, equivalently a SPCS*, if, and only if, $X = \{z^*\}$ for some Pareto efficient state $z^* \in Z$. Intuitively, regardless of the other players' actions, any coalition can threaten with a swinging final state in a farsighted normal form game if it so desires, simply by always electing to change the current state whenever it is selected to make a choice. Such a threat can be used instead of a myopic equilibrium within the strategy profile supporting a SPCS / SPCS*. We will now show that this result can be strengthened even further: consider an extension of Definition 2.1 allowing swinging, w, to be a final state and a member of a SPCS / SPCS*, where w is assigned some utility u_i^w (finite or $-\infty$) for player i. Indeed, as argued above, in a generic normal form game, regardless of the other players' actions, any coalition can force a swinging final state in a farsighted normal form game if it so desires. The definitions of $\Psi_{\sigma|h}$ and $d_{\sigma|h}$ are now extended to any strategy profile σ, coinciding with $\Psi_{i,\sigma|h}$ and $d_{i,\sigma|h}$ for all i. Note that Proposition 2.1 still holds: a SPCS / SPCS* is always non-empty. This leads to the following result.

Theorem 3.2 *For any generic normal form game, $X \subseteq \bar{Z}$ is a SPCS, equivalently a SPCS*, if, and only if, $X = \{z^*\}$ for some Pareto efficient state $z^* \in \bar{Z}$ such that $u_i(z^*) \geq u_i^w$ for* *all $i \in N$.*

Theorem 3.2 provides us with necessary and sufficient conditions for the existence of a SPCS / SPCS* when swinging is allowed and its normalized utility is zero for all players. Reflecting back on Example 3.3 we can appreciate the effect of allowing a swinging behavior.

Example 3.4 *Farsighted Matching Pennies with swinging assigned utility $u_i^w = 0$. By Theorem 3.2, the singleton set $\{w\}$ consisting of a swinging final state is the unique SPCS. This is true because w is the unique Pareto efficient state in $\bar{Z}$ providing all players with non-negative utilities (note that any other state is also Pareto efficient but fails the non-negativity condition, thus it is not reachable from itself).*

3.1 Related Literature on Stability in Normal Form Games

Greenberg (1990) was the first to study myopic stability in non-cooperative games. He has characterized the (myopic) vNM solution in the 2-player Prisoners' Dilemma, and, for example, has proven existence of the (myopic) vNM solution for any 2-player normal form game with finite strategy sets. The vNM FSS in normal form games was first studied by Muto (1993), who has shown that in the Prisoners' Dilemma problem, the vNM FSS coincides with the (myopic) vNM solution[8]. Suzuki and Muto (2005) have shown that in the class of n-person Prisoners' Dilemma games, with coalitional moves, any individually rational and Pareto efficient outcome is a vNM FSS and no other vNM FSS exists. Kawasaki (2015) and Bloch and van den Nouweland (2021) studied the class of two-person normal form games with finite strategy sets. They proved that, with pairwise moves, any strictly individually rational and Pareto efficient strategy profile forms[9] a singleton vNM FSS. They have further characterized the vNM FSS for all two-person normal form games with finite strategy sets. They have shown, for example, that the Prisoners' Dilemma problem is the only two-person normal form game that does not have a singleton vNM FSS. Indeed, it follows from Proposition 5.1 in Bloch and van den Nouweland (2021), as well as from Proposition 3.7 therein, and Muto (1993)'s result mentioned above, that the vNM FSS in the Prisoners' Dilemma problem contains the Nash equilibrium. By comparison, we have shown in Example 3.1 that in the Prisoners' Dilemma problem, the SPCS consists uniquely of the pair of strategies yielding the cooperative (socially optimal) solution.

[8] For related studies which investigate the vNM FSS in more general normal form games than the Prisoners Dilemma, see, e.g., Suzuki and Muto (2005), who allow for coalitional deviations, and Nakanishi (2009), wherein only individual deviations are possible.

[9] Kawasaki (2015)'s proof had to invoke a mild restriction on the payoffs.

We note that although the SPCS and the vNM FSS do not coincide in the Prisoners' Dilemma problem, they do coincide in some other instances of two-person normal form games. For example, Bloch and van den Nouweland (2021) have proven that, for two-person normal form games, (i) if a strategy profile s is a Nash equilibrium that is not Pareto dominated by any other strategy profile, then $\{s\}$ is a singleton farsighted stable set (Corollary 4.9 therein), and (ii) if s Pareto dominates all other strategy profiles, then $\{s\}$ is a singleton farsighted stable set and it is also the unique farsighted stable set (Corollary 4.10 therein). Then, it can be shown that our Theorem 3.1 implies that in case (i), $\{s\}$ is also a singleton SPCS for n-person normal form games, and that in case (ii), $\{s\}$ is also the unique SPCS for n-person normal form games.

D&V (2020) have introduced a three-country pollution abatement game in which each country i has two strategies, denoted as $x_i \in \{0, 1\}$, representing low cost and high cost abatement technology strategies, respectively. For a given technology strategy vector by the three countries, (x_1, x_2, x_3), the utility for country i is given by $u_i(x_1, x_2, x_3) = x_i - \frac{1}{2}(\sum_{j \neq i} x_j)^2$. They modelled the pollution abatement problem as a normal form game in which the three countries select their abatement technologies simultaneously. In this normal form game formulation, $x_i = 1, i = 1, 2, 3$, is the unique Nash equilibrium, while the set of strategy vectors which maximize the sum of the payoffs to the three countries is $T = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Then, it was proven by D&V (2020) that their history dependent farsighted solution concept, HREFS, consists of T, while T cannot be sustained as the history independent version of HREFS, REFS, introduced by D&V (2017)[10]. Clearly, since T is also the set of Pareto efficient strategy vectors dominating the Nash equilibrium, it follows from Theorem 3.1 that each strategy vector in T is a singleton SPCS for the three-person pollution abatement problem.

3.2 Some Applications

In this section we briefly illustrate the application of the SPCS approach to analyze various classical oligopolistic settings, and some further examples.

3.2.1 Farsighted Bertrand and Cournot Competition

In the Bertrand and Cournot examples discussed in this subsection, as is the case in, e.g., Xue (1998), Masuda et al. (2000) and Suzuki and Muto (2006), the players "compete" only once, based on the final quantities or prices reached. Intermediate quantities or prices are transient states in a tacit negotiation. They are not offered to the consumers and they

[10] See definitions of these two solution concepts in Section 4.

bear no utility consequences whatsoever to the players. Thus, only the final values matter. Swinging should be avoided, since if it occurs, revenues that could be realized by the players will be forfeited. In the following we show that the SPCS extends the efficiency results obtained in the literature for symmetric settings using vNM FSS (Masuda et al. 2000).

Farsighted Bertrand Competition Consider a market with n sellers of a homogeneous product that are engaged in farsighted price competition, in which seller i sets price p_i. The sellers face a downward sloping demand function $D(p)$, where p is the lowest among the prices they set. Suppose that the sellers have equal sales power, so that when the same price is set by several sellers, each of them sells the same quantity. Assume further that there are no fixed costs and the unit cost of the product for seller i is c_i, with $\bar{c} \equiv \min_i c_i$. When the unit costs are not all equal there is no real price competition because the seller with the lowest unit cost can always set the price slightly below the second lowest unit cost and gain the entire market alone.

It is well known that when this situation is viewed as a normal form game, there is a unique Nash equilibrium in prices: all sellers with cost $\bar{c}$ set the price $p^* = \bar{c}$ and the market is split between them equally with zero profit to each of them.

For this setting a SPCS is $X = \{z^*\}$ where z^* is a Pareto efficient state in which several (possibly all) sellers with cost lower than the monopolistic price, i.e. the price p that solves

$$\max_p (p - c) D(p),$$

each sets this price, which allows them to share the monopolistic profit equally among them. The remaining sellers (if any) set some price higher than the monopolistic price and receive zero profits. This follows from Theorem 3.1 because such states z^* are the only Pareto efficient states in Z that weakly Pareto dominate the unique Nash equilibrium state. For comparison, Masuda et al.(2000) show for symmetric settings that the LCS contains all states with positive profits to all firms, including Pareto non-efficient states.

Farsighted Cournot Competition Consider a market with n sellers of a homogeneous product that are engaged in farsighted quantity competition, in which each seller i sets quantity q_i. The sellers face a decreasing inverse demand function determining the market price as, for simplicity, $\max\{a - Q, 0\}$, where $Q = \sum_i q_i$ is the total quantity sold. Suppose that there are no fixed costs and the unit cost of the product for seller i is c_i. Thus the profit to seller i is $\pi_i \equiv q_i(a - Q - c_i)$ when $Q \leq a$ and zero otherwise.

It is well known that this normal form game has a unique Nash equilibrium in quantities, in which player i sells quantity

$$q_i^* = a - c_i - \frac{1}{n+1} \sum_j (a - c_j)$$

and makes profit $\pi_i^* = (q_i^*)^2$ (this holds under the assumption that $a \geq (n+1) \max_j c_j - \sum_j c_j$, otherwise the market is not large enough to include all sellers in equilibrium). In the symmetric case in which $c_i = c$ for all i, each player sells in equilibrium $\frac{a-c}{n+1}$ and makes profit $(\frac{a-c}{n+1})^2$.

To find the SPCS solution for this setting, Theorem 3.1 says that we need to compute the Pareto efficient states that weakly Pareto dominate the unique Nash equilibrium state. Since in any state $z \in Z$, the quantity sold $q_j = \frac{\pi_j}{a-c_j-Q}$ for all j, the profit π_i of any seller i can be written as a function f of the total quantity sold and the other sellers' profits, given by

$$f[Q, (\pi_j)_{j \neq i}] \equiv (Q - \sum_{j \neq i} \frac{\pi_j}{a - c_j - Q})(a - c_i - Q).$$

Therefore a Pareto efficient vector of profits, for which $\pi_j \geq \pi_j^*$ for all $j \neq i$, must satisfy that π_i maximizes $f[Q, (\pi_j)_{j \neq i}]$ over the variable Q and subject to the constraint $\pi_i \geq \pi_i^*$.

In the symmetric case, $f[Q, (\pi_j)_{j \neq i}]$ simplifies to $Q(a - c - Q) - \sum_{j \neq i} \pi_j$, for which the maximal Q equals $\frac{a-c}{2}$, independently of the other sellers' profits. In this case seller i's profit is $\pi_i = (\frac{a-c}{2})^2 - \sum_{j \neq i} \pi_j$, as long as $\pi_j \geq \pi_j^*$ for all j (including i), and the total profit is $(\frac{a-c}{2})^2$. Such Pareto efficient states always exist because for all $n \geq 2$, the total profit is always higher than the total Nash equilibrium profit, $n(\frac{a-c}{n+1})^2$. This allows the total profit increase to be shared in some way between the sellers. Therefore the SPCS reasoning implies that the sellers would share between them a monopolistic total quantity, allowing them to share a monopolistic total profit in a way that improves for each of them on the Nash equilibrium profit. For comparison, Masuda et al.(2000) show for symmetric settings that the LCS contains all states with non-negative profits to all firms, including Pareto non-efficient states.

3.2.2 Decentralized Supply Chain Contracting

The SPCS can be used to analyze decentralized supply chains and can be shown to lead to full coordination and Pareto efficiency. We demonstrate this general point in the classic setting of a vertical decentralized supply chain, based on the newsvendor model, with a single supplier and several competing retailers. As is well known, system coordination and Pareto efficiency are trivially achieved in a cooperative bargaining/negotiation modelling of this setting (see, e.g., Nagarajan and Bassok 2008). We demonstrate that they can also be achieved in a non-cooperative modelling of this problem when players are assumed to be farsighted. Moreover, coordination is achieved contrary to the inability of revenue sharing contracts to do so in general (see Cachon and Lariviere 2005).

The market consists of n retailers that face uncertain demand for a differentiated product, where retailer i's demand $D_i(p_1, \ldots, p_n, \varepsilon)$ is decreasing in their own unit price p_i, increasing in each of the other retailers' unit prices p_j and depends on a common random variable ε. Each retailer decides about the price p_i and the inventory level q_i to be ordered from a single supplier before demand realization. The supplier charges each retailer a wholesale price w per unit ordered and a share, m, of the retailer's total revenue. The unit cost for the supplier is c with no fixed costs.

When this arrangement is viewed as a revenue sharing contract signed between the supplier and the retailers, the setting becomes a two stage model: first the supplier sets the charges, (w, m), under equal[11] contractual terms across the retailers, and then the retailers set the prices, p_i, and order the quantities, q_i. Solving for Nash equilibrium of the second stage, each retailer's profit, $(1 - m)p_i E_\varepsilon \min\{q_i, D_i(p_1, \ldots, p_n, \varepsilon)\} - wq_i$ (where E_D is the expectation operator over the distribution of the random variable ε), is maximized given the other retailers' choices, leading to equilibrium prices and quantities as a function of the contract parameters (w, m). In the first stage one seeks a contract that coordinates the supply chain by achieving the system optimal quantities and prices. Note that such revenue sharing contracts, when the retailers must be offered equal contractual terms by the supplier, are unable to coordinate the system when the retailers are asymmetric (see, e.g., Cachon and Lariviere 2005 and Krishnan and Winter 2011).

The implications of the SPCS reasoning for this setting are very different. A state z in the game consists of a vector $[w, m, (p_1, q_1), \ldots (p_n, q_n)]$, so the wholesale price, the revenue share m, the inventory levels q_i and the prices p_i are viewed as alternatives than can be altered in the game tree without limit. By Theorem 3.1, we look for the Pareto efficient states that weakly dominate the Stackelberg equilibrium. The wholesale price w and the revenue share m do not affect the system profit, but only determine how it is shared between the players. Therefore a SPCS is a single state z^* consisting of optimal prices and quantities for the system, and wholesale price and revenue share values that split the optimal system total profit such that each player is weakly better off than in the equilibrium. Such wholesale price and revenue share values exist whenever the equilibrium system profit is strictly lower than the optimal one. Therefore, in contrast to the traditional equilibrium analysis, the SPCS reasoning leads to system coordination.

[11] Equal contractual terms are legally required in the USA by the Robinson–Patman Act of 1936.

3.2.3 Farsighted Network Formation

Networks are often used to model interactions among individuals in various social and economic situations, see, e.g., Jackson (2010). The individuals, referred to as players, are represented by nodes in a graph and a link, or edge, (i, j), in the graph stands for an existing interaction between players i and j. Players strive to maximize their utilities from interactions with other players, and for that purpose, establish links or dissolve existing ones. The two main questions addressed in the literature are which networks, to be referred to as stable networks, could emerge from such a process and whether stable networks are efficient, i.e., maximize the total utility associated with the interactions among all players. In static models of network formation, the players derive their utilities only from the final network that has been reached, i.e., a stable network.

Evidently, there is a tension between stability and efficiency. That is, stable networks are not necessarily efficient. In the static case, this result was shown by Jackson and Wolinsky (1996) for the myopic case, and for the farsighted case by Herings et al. (2009). In the dynamic case, where players get discounted utilities during the entire network formation process, the tension between efficiency and stability was confirmed by Dutta et al. (2005). By contrast, however, Kimya (2020) has proven, for example, that in the dynamic case, under some conditions, every efficient network can be supported as the prediction of his farsighted solution concept, (C)ECB. Finally, Luo et al. (2021), who have used the *myopic-farsighted stable set* to study network stability with both myopic and farsighted players, have shown that, under some conditions, replacing myopic players with farsighted players alleviate the tension between stability and efficiency.

We propose to model network formation as a farsighted 1-normal form game. For each player i, the set of alternatives, A_i, consists of all coalitions in $N \setminus \{i\}$, and player i's action, $S \subseteq N \setminus \{i\}$, represents the set of players/nodes with whom she would like link. The network, at the outset of any stage k, consists of all edges, (i, j), for which players i and j, in their most recent moves prior to stage k, if any, have both included one another among those players with whom they want to link. Note that existing network formation processes studied in the literature involve coalitions move, mostly by pairs of players who form links between themselves. In the network formation process embodied in the farsighted 1-normal form game, only individual players are involved.[12]

To present our result, we need to introduce some basic definitions. An allocation rule associated with a network g allocates the value of g, $v(g)$, among the players. It is commonly assumed that the value function $v(g)$ is *component additive*, that is, $v(g)$ is equal to the sum of

[12] For a related network formation literature in which individual players, unliterally, establish costly links to access benefits generated by other players, see, e.g., Bala and Goyal (2000).

the values of its maximally connected components, and that an allocation rule is *component balanced*, meaning that for each maximally connected component, C, of g, the value, $v(C)$, is divided among the players in C.

Now, it is usually assumed that $v(\{i\}) = 0$ for all i. Indeed, Dutta et al. (2005) have normalized the value of singletons to zero, and for some network models studied in the literature, see, e.g., Jackson and Wolinsky (1996) and Dutta et al. (2005), the values of singletons implied by the context of the applications are zero. So, without much loss of generality, we will assume that the values of singletons are zero. Finally, note that for non-negative value function and allocation vector, there is a Nash equilibrium in the myopic normal form game associated with network formation, wherein the utility for each player is zero.

Then, our contributions to mollifying the tension between efficiency and stability in network formation, achieved only by actions taken by individual players, follows from part (2) of Theorem 3.1:

Corollary 3.1 *Assume a non-negative and component additive value function and component balanced and non-negative allocation vector, and consider the farsighted* 1*-normal form game representing a network formation process carried only by individual players. Then, (i) a network g is Pareto efficient if and only if $\{g\}$ is a* SPCS *or* SPCS**, in particular, if g is a socially efficient network then $\{g\}$ is a* SPCS *and* SPCS**, and (ii) if there exists a network g that strictly Pareto dominates all other networks, then $\{g\}$ is the unique* SPCS *and* SPCS**.*[13]

4 Comparison to the Related Literature

We briefly review in this section related farsighted solution concepts introduced in the literature, and use Example 1.1 discussed in the Introduction, as well as other examples, to compare the SPCS to these solution concepts.

The SPCS, as well as many other related solution concepts, e.g., those introduced by Chwe (1994), M&V (2004), D&V (2017), D&V (2020), and K&R (2021), are defined in a general setting, described by an abstract game, as introduced in Section 2. Indirect dominance, first proposed by Harsanyi (1974) in the context of coalitional games, and then employed by Chwe (1994) to introduce the Largest Consistent set (LCS) for abstract games, is defined as follows:

[13] See a similar result by Herings et al. (2009) and their pairwise farsightedly stable set.

Definition 4.1 *[Harsanyi 1974, Chwe 1994]. Indirect Dominance: We say that a state* $a \in Z$ *is indirectly dominated by a state* $b \in Z$*, or* $b \gg a$*, if there exist states* $a_0, a_1, \ldots, a_m \in Z$*, where* $a_0 = a$ *and* $a_m = b$*, and coalitions* $S_0, S_1, \ldots, S_{m-1}$*, such that* $a_l \rightarrow_{S_l} a_{l+1}$ *and* $u_i(b) > u_i(a_l)$ *for all* $l = 0, 1, \ldots, m-1$ *and* $i \in S_l$*.*

Definition 4.2 *[Consistent Set, Chwe (1994)]. A set* K *is consistent if* $K = \{x \in Z \mid \forall y, S$ *with* $x \rightarrow_S y$*,* $\exists z \in K$*, where either* $z = y$ *or* $z \gg y$*, such that* $u_i(x) \geq u_i(z)$ *for some* $i \in S\}$*.*

The LCS employs a pessimistic, rather than an optimal criterion for a move. That is, a move by coalition S from a State z is rejected if it could lead, via indirect dominance, to a stable state at which not all members of S are strictly better off. Nevertheless, the LCS has remained an important solution concept that has received much attention in the literature, and its relation to other solution concepts are often explored.

The vNM solution for abstract games, wherein ordinary dominance is replaced by indirect dominance, referred also as the vNM farsighted stable set (vNM FSS), is contained in the LCS (Chwe 1994). It leads to sharper results than the LCS in several instances, such as symmetric Cournot and Bertrand oligopoly markets (Suzuki and Muto 2006 and Masuda et al. 2000), and one-to-one matching games (Mauleon et al. 2011 and Diamantoudi and Xue 2003), and it was further shown to generate new insights into patent licensing negotiation (Hirai et al. 2019). However, the vNM FSS could still yield not very insightful results. For example, for a game based on the provision of a perfectly 'lumpy' public good, without coalitions, it was shown by Kawasaki and Muto (2009) that the vNM FSS includes almost all individually rational outcomes.

Page and Wooders (2009) have introduced a model of network formation whose primitives consist of a set of networks, players' preferences, rules of network formation, and a dominance relation on feasible networks. As noted by the authors, their network formation game can be viewed as an abstract game with a finite set of states and a dominance relation among states which could be either direct dominance, indirect dominance, or path dominance.

It should be noted that, prior to Chwe, Greenberg (1990) has developed the theory of social situations, wherein he has introduced the concept of a stable standard of behavior (SB), which is claimed to capture perfect foresight by individuals or coalitions. Some precise relationships between a stable SB and the LCS were investigated by Chwe (1994). Xue (1998) has argued that the LCS captures only partial foresight, and by employing Greenberg's (1990) framework, characterized a set of paths which constitutes a stable SB. See also Brams (1994)'s theory of moves (also Brams and Wittman 1981, Kilgour 1984) for a dynamic, finite game tree approach to two-player, two-strategy normal form games, and Mariotti (1997), wherein a farsighted solution concept similar to a stable SB is shown to achieve partial efficiency in normal form games with coalitions.

The pessimistic criterion embedded in the definition of the LCS is somewhat mollified in Mauleon and Vannetelbosch (M&V, 2004) Largest Cautious Consistent set (LCCS). The LCCS is a proper refinement of the LCS, which could possibly be empty, wherein players are less pessimistic than in the LCS. Specifically, as argued by M&V, a state is never farsighted stable if a coalition, S, can move therefrom, and by doing so there is no risk that some coalition members of S will end up worse off, while such a move could also lead to a stable state at which some or all members of S being strictly better off. The LCCS does provide new insights in some instances, as shown, e.g., by M&V (2004) in their study of coalition formation, and by Granot and Yin (2008) in their analysis of a single-period two-stage supply chain problem.

This criticism against the LCS, LCCS, and vNM FSS has led to the development of other farsighted solution concepts for abstract games, see, e.g., D&V (2017), D&V (2020), Kimya (2020) and K&R (2021). These new farsighted solution concepts incorporate various maximality requirements for coalitions' moves, and they further adopt the logic underpinning the vNM solution and satisfy vNM type internal and external stability requirements. They employ a modification of Jordan's (2006) expectation function and introduce an expectation function, F, to describe the transition from one state to another, as well as the coalition that is supposed to affect that transition in the abstract game. The RE function F ensures that all players have commonly held beliefs about the sequence of coalitional moves, if any, from every state.[14] Thus, F predicts with certainty the unique coalition, S, to be active at any state and the derived state, after the move by S, if any. If no coalition wants to change the current State, z, then z is referred to as a stationary state of F. An expectation function, F, is said to be absorbing if for every $z \in Z$, the unique path prescribed by F leads to a stationary point in Z.

An absorbing RE function F is said to be rational if it satisfies vNM type of internal and external stability requirements, as well as a maximality requirement, in the sense that the move by any coalition from a non-stationary point must be maximal. That is, there does not exist another move leading to another stationary point at which all members of the moving coalition are strictly better off. As noted by D&V (2017), their internal stability requirement is weaker than the ordinary vNM internal farsighted stability requirement, since the farsighted objection has to be consistent with F. For the same reason, their external stability requirement is stronger than the ordinary vNM external farsighted stability requirement.

D&V (2017) have proposed the set of all stationary points of a rational expectation F as a

[14] See Bloch and van den Nouweland (2020), for farsighted stability with heterogeneous expectations.

farsighted stable set and refers to it as the rational expectation farsighted stable set (REFS). They have further introduced the strong rational expectation farsighted stable set (SREFS), which consists of all stationary points of a rational expectation F, in which the maximality requirement is replaced by a stronger requirement, referred to a strong maximality. As shown by D&V (2017), in general, both REFS and SREFS may not exist, but they were shown to exist for some classes of games.

Dutta and Vartiainen (2020) have extended D&V (2017) to history dependent rational expectation functions, and have proposed the history dependent rational expectation farsighted stable set (HREFS) and the history dependent strong rational expectation farsighted stable set (HSERFS) solution concepts, which are the history dependent analogues of REFS and SREFS, resp. A history at State z is the sequence of past moves and the coalitions involved in these moves until State z was reached. A history dependent expectation function specifies the active coalition and its move for all possible current states and past histories. Clearly, history independence is a special case of history dependence. Thus, as noted by D&V (2020), REFS and SREFS, are special cases of HREFS and HSREFS. Indeed, D&V (2020) were able to derive non-emptiness results for their history dependent farsighted solution concepts, which are not available for the REFS and SREFS solution concepts.

We note that in the rational expectation function solution concepts introduced by D&V (2017), REFS and SREFS, as well as those introduced by D&V (2020), HREFS and HSERFS, progression in the associated abstract game specified by a RE function F is done along indirectly dominated paths. That is, for a given RE function F, or RE F with an associated history, H, if F specifies a path, p, from some state z to a stationary state z^*, then the members of all the coalitions involved with the progression from z to z^* strictly prefer the utilities they derive at z^* than those they attain at the states on p wherefrom they moved according to F.

Karos and Robles (2021) have pointed out that when coalitions are provided with an opportunity to move from a State z, they should be farsighted enough so as to compare the consequences of their move from z to remaining at z and let other coalitions move from z. Indeed, K&R (2021) introduced the extended expectation function, F, which assigns to each state z, an ordered list $(F^1(z), \dots, F^{k(z)}(z))$, where F_1 is the D&V (2017) basic expectation function that specifies the transitions among states, and each $F^j(z)$ consists of a coalition, $S^j(z)$, and a unique state, $f^j(z)$, to which coalition $S^j(z)$ can move if all previous coalitions on the ordered list, $S^k(z)$, $k = 1, \dots, j-1$, have elected not to move from z.

A rational extended expectation function (REEF) is an extended expectation function that, similar to REFS, SREFS, HREFS, and HSREFS, satisfy vNM type internal and external stability constraints, as well as satisfying the (ordinary) maximality constraint introduced by D&V (2017).

Specifically, for each $z \in Z$ let $S^1(z), \ldots, S^{k(z)}(z)$ be the ordered list of coalitions at State z, so that coalition $S^l(z)$ will get to move from state z to State $f^{l(z)}$ if all previous coalitions on the ordered list elected not to move from z. Then K&R (2021) have imposed the following requirements from their rational extended expectation function:

Internal Stability (I). For all $z \in Z$ and all coalitions $T \notin \{S^1(z), \ldots, S^{k(z)}(z)\}$ there is $l \leq k(z)$ such that for each $y \in Z$ with $z \to_T y$ there is $i \in T$ for which $u_i(f^l(z), F) \geq u_i(y, F)$.

External Stability (E). For all $z \in Z$ and for all $l = 1, \ldots, k(z) - 1$, it holds that $u_i(f^l(z), F) > u_i(f^{l+1}(z), F)$ for all $i \in S^l(z)$.

Maximality (M). For all $z \in Z$ and for all $l = 1, \ldots, k(z) - 1$ it holds that if there is $y \neq f^l(z)$ such that $z \to_{S^l(z)} y$, then there is $i \in S^l(z)$ for whom $u_i(f^l(z), F) \geq u_i(y, F)$.

A rational extended expectation function (REEF) is an extended expectation function that satisfies the I , E , and M requirements.

As noted by K&R (2021), their internal and external stability constraints are stricter than those imposed by, e.g., D&V (2017), since coalitions/players in their model also need decide whether to move or not. The set of all stationary points of a rational extended expectation function is an equilibrium stable set (ESS). If it exists, the ESS is not empty. However, similar, e.g., to the vNM stable set, vNM FSS, and REFS, the ESS may not exist, as is the case, e.g., in the farsighted roommate problem.

Kimya (2020) has studied farsightedness in the class of extended coalition games. An extended coalition game is closely related to an abstract game, with the important exception that in an extended coalition game, the utilities of players are defined over the paths of play rather than on terminal states, which allows the model to accommodate both the static and dynamic approaches to farsightedness. Kimya employs the concept of a coalition behavior, which, like an expectation function, provides a complete plan of action that assigns a unique action to each state, to ensure that players have commonly held beliefs about the coalitional moves at any state. Thus, it prescribes a unique terminal path to each node of an extended coalitional game, where a terminal path is either infinite or ends with a terminal state, wherefrom there is no move.

Kimya (2020) has introduced two related solution concepts, the equilibrium coalition behavior (ECB) and the credible ECB, CEBC, and has shown, for example, that CECB exists in any finite extended coalitional game. He has further explored the relationships between his solution concepts and other related solution concepts in various classes of games, such as, e.g., farsighted network formation games briefly covered in Sub-Section 3.2.3.

Let us next analyze several simple examples of farsighted abstract games, and compare the predictions of the various farsighted solution concepts of these examples.

First, recall Example 1.1. As discussed in the Introduction, and further elaborated on in Section 2, the unique SPCS of the abstract game associated with Figure 1.1 is {D,E}, and as such, it awards Players 1 and 2 utility of either 1 or 3.

Let us next study the other farsighted solution concepts for Example 1.1, starting with the LCS. Again, clearly, as previously mentioned, States D and E are contained in the LCS, as well as in all other farsighted solution sets. Further, State E indirectly dominates States A,C, and no other indirect dominance relation holds in this example. Since a move by Player 2 from State B to State C could end up only at State E at which Player 2 is not strictly better off, State B also belongs to the LCS. In contrast, State A is not in the LCS since a move by Player 1 from State A can only end up at State E at which Player 1 is strictly better off. We conclude that in Example 1.1, LCS = {B,D,E}. Similarly, since also according to the LCCS there is complete certainty in this example regarding the state at which any move may end up, the LCCS = {B,D,E} too. Finally, since State E indirectly dominates State A, State A is not contained in the vNM FSS. However, State B is not indirectly dominated by any other state, and thus, State B is in the vNM FSS, and we conclude that the vNM FSS of Example 1.1 is also {B,D,E}.

By comparison, let us apply the RE approach, REFS and SREFS (D&V 2017), HREFS and HSREFS (D&V 2020), to the above example. According to this approach, a moving coalition at each state knows with confidence the final state her move would lead to, and all coalitions on the path in the abstract graph leading to the final state are strictly better off at the final state than at the state wherefrom they moved. Player 3 will not move from State C since they are strictly worse off at State D than at State C. Player 4 is strictly better off moving from State C to State E, therefore moves by Players 1 and 2 to State C would end up at State E, where Player 1 is strictly better off than at States A, while Player 2 is just equal off. Then, in this case, Player 1 (resp., 2) will move (resp., stay) at State A (resp., B), leading to {B,D,E} as a stable set. We note that history dependence, as allowed by D&V (2020), does not alter the conclusion.

According to the K&R (2021) modification of the RE function approach, with each state, z, there is an associated ordered list of the effective coalitions that get to move at State z. So, in the above example, if Player 4 gets to move first at State C, they will elect to do so, which would lead to {B,D,E} as a stable set. However, if Player 3 gets to move first at State C, then, in contrast with the solution discussed in the previous paragraph, they will do so, to preempt a later move by Player 4 to State E, at which Player 3 is strictly worse off than at State D. Indeed, K&R (2021) argue that farsighted players should strictly prefer to carry out a pre-emptive move from some state z, which would lead to a stable state, say, y, at which they are strictly worse off, rather than let another coalition move from state z, which would end up at another stable state at which they are even strictly worse off than at state y. But, at State D, Player 1 is equal off than at State A and Player 2 is strictly better off compared to State B. Thus, if Player 3 gets to move first at State C, Player 1 will not move from State A, while Player 2 will move from State B, yielding {A,D,E}, as another stable set according to K&R (2021) modification of the RE function approach. We conclude that the stable sets according to the RE solution concepts predict the sets {A,D,E} and {B,D,E}.

Next, let us consider the prediction of Kimya's (2020) static solution concepts, (C)ECB, for Example 1.1. Thus, we assume that payoffs are realized only at the final states. Consider Case (i), where coalition behavior, ϕ^1, prescribes a move by Player 4 at State C, and moves by Players 1 and 2 at States A and B, respectively. Then, neither Player 1, nor Player 2, nor Player 4 has a profitable deviation from ϕ^1. Thus, ϕ^1 is a (C)ECB, leading to a stable set {D,E}. In Case (ii) consider the coalition behavior, ϕ^2, where Players 1 and 4 move but Player 2 stays at State B. Again, the players do not have a profitable deviation from ϕ^2, yielding a stable set {B,D,E}. A similar analysis, where Player 3 moves at State C with no credible profitable deviation, would yield a CECB and the stable sets {A,D,E} and {D,E}, and we conclude that the static CECB solution concept yields the stable sets {A,D,E}, {B,D,E} and {D,E}.

Thus, farsightedness, as embodied in all solution concepts other than the SPCS, stipulates that either Player 1 or Player 2 may forgo a non-deterministic prospect of utilities $1, 3$, and remain at either State A or B, respectively, at which they will attain with certainty a utility of 1. Note that, in part, the reasoning behind the preference within the SPCS for a non-deterministic prospect of utilities $1, 3$ is in the spirit of the LCCS modification to the LCS, which, by contrast with the LCS, prescribes a move when it may lead to strictly better states and to no worse states. Indeed, the SPCS, as per Assumptions 2.1 and 2.2, is guided by first order stochastic dominance to decide whether a state wherefrom a move is initiated is stable. Nevertheless, the SPCS delivers here the intended prediction, whereas the LCCS fails to do so.

Next, recall the farsighted Prisoners' Dilemma (PD) game analyzed in Section 3, with only individual moves. In the abstract game representation of the PD, Player 1 (row player) can move between States (C,C) and (D,C) and between States (C,D) and (D,D), while Player 2 (column player) can move between States (C,D) and (C,C) and between States (D,C) and (D,D). The following matrix describes the indirect dominance $\gg$ for this game, indicating the cells for which the column state is indirectly dominated by the row state. For comparison, our reachability relations R, R^* are also indicated in a similar manner.

	(C,C)	(C,D)	(D,D)	(D,C)
(C,C)	R R^*	R R^*	R R^*	R R^*
(C,D)	$\gg$			
(D,D)	R R^*	$\gg$ R R^*	R R^*	$\gg$ R R^*
(D,C)	$\gg$			

As can be seen from the above table, the state corresponding to (D,D) is not indirectly dominated by any other state, the states corresponding to (C,D) and (D,C) are indirectly dominated by (D,D), and the state corresponding to (C,C) is not indirectly dominated by (D,D). We can therefore conclude that the LCS, LCCS, vNM FSS, and the RE solutions concepts introduced by D&V (2017) and D&V (2020), in which progression in the abstract game is along indirect dominating paths, consist of the cooperative and non-cooperative strategy pairs, {(C,C),(D,D)}. Further, note that in the PD example, the utilities of each of the two players in the four states are distinct. Then, since we do not consider non-singleton coalitions, we can use Proposition 6 in Kimya (2020) to conclude that the SREFS {(C,C),(D,D)} is also the stable set corresponding to the static ECB (and thus is a stable set corresponding to the static CECB) of the farsighted PD game. Finally, we also note, via the next result, that this is also the unique ESS for this example.

Lemma 4.1 *The unique ESS for the PD game example is* {(*C,C*),(*D,D*)}.

In contrast, as shown in Section 3, the unique SPCS for this game is {(C,C)}. Some intuition for this stark difference may be gained by comparing, via the above table, the relative strength of subgame-perfect-reachability, as compared to indirect-domination-reachability. Specifically, as mentioned earlier, according to indirect-domination-reachability, the only reachable states from an initial state are those states which indirectly dominate it. By contrast, in the SPCS, players are guided by subgame perfection, which allows them to reach states, from some starting state, which are not necessarily indirectly dominating it. For example, in the PD example, the state corresponding to (C,C) is subgame-perfect-reachable from, e.g., the state corresponding (D,D), but it is not indirect-domination-reachable from (D,D).

In fact, the drivers of and the intuition behind the proof of Theorem 3.1 are the strength of the subgame-perfect-reachability relation and the consistency requirement satisfied by the SPCS. Specifically, as demonstrated therein, the proof follows since (i) any state in the SPCS of a farsighted normal form game is surely reachable from any state in Z, and (ii) off the equilibrium path, any deviation to some new state could lead, with positive probability, to any state in the SPCS.

To summarize, in the farsighted PD example, LCS = LCCS = vNM FSS = REFS = ESS = {(C,C), (D,D)}, and the SPCS = {(C,C)}. Thus, in the farsighted PD, being a SPCS neither implies nor is implied by being any of the other farsighted solution concepts sets.

Next, consider Example 1.1', represented by Figure 1.1 with the modified utility vectors $(3,3,0,0),(3,3,0,0),(0,0,3,0),(0,0,2,0)$ and $(4,4,0,3)$. Now, the only indirect dominating paths in Example 1.1' are from State A to State E, and from State B to State E. Since both Players 1 and 2 are strictly better off at State E than at State A and B, respectively, States A and B are not contained in the LCS. That is, LCS = {D,E}. Similarly, we also have LCCS = vNM FSS = {D,E}. Moreover, since according to the RE solution concepts, progression dictated by the RE function is carried out only along indirect dominating paths, these RE solution concepts also lead to {D,E}. However, according to the ESS (K&R 2021), progression in the abstract game, as dictated by the associated RE function F, is not necessarily carried out along indirect dominating paths. Indeed, following the same logic discussed in the context of Example 1.1, according to the ESS solution concept, Player 3, given an opportunity to move at State C, prefers to move to State D, at which their utility is reduced from 3 to 2, because if they choose not to move and let Player 4 move from State C to State E, their utility will be zero. We note that at State D, Players 1 and 2 are strictly worse off than at States A and B, respectively. Therefore, if Player 3 is the first to move at State C, Players 1 and 2 will not move from A and B, respectively. If, however, Player 4 is the first to move at State C, then she will move to State E, at which both Players 1 and 2 are strictly better off than at States A and B, respectively. We conclude that the ESS of Example 1.1' are either {A,B,D,E} or {D,E}.

Next, let us consider the prediction of Kimya's (2020) static solution concepts, (C)ECB, for Example 1.1'. Again, we assume that payoffs are realized only at the final states. Consider Case (i), where coalition behavior, ϕ^1, prescribes a move by Player 4 at State C, and moves by Players 1 and 2 at States A and B, respectively. Then, as in Example 1.1, all three players don't have a profitable deviation from ϕ^1. Thus, ϕ^1 is a (C)ECB, leading to a Stable Set {D,E}. In Case (ii) consider the coalition behavior, ϕ^2, where Player 3 moves, and Players 1 and 2 do not move. As in Example 1.1, Player 3 does not have a credible deviation from ϕ^2. Further, Players 1 and 2 do not have a profitable deviation from ϕ^2, leading to the Stable Set {A,B,D,E}.

Let us consider the SPCS of Example 1.1'. Recall that if Player 3 is given an opportunity to move at State C, they will prefer to move to State D. Further, if Player 4 is given an opportunity to move at State C, they will elect to do so since they are better off at State E than at State C and at State D. Thus, according to the SPCS, there is a subgame perfect equilibrium in which, moves from States A and B could end up at State D (and E). Since both Players 1 and 2 are strictly worse off at State D, and strictly better off at State E, than at States A and B, then, using, e.g., expected utility as the optimality criterion, for a range of endogenously determined probabilities for which Player 3 is the first to be granted the opportunity to move at State C, Players 1 and 2 would prefer not to move from States A and B, respectively. For a different range of probabilities, Players 1 and 2 would prefer to move from States A and B. Since these probabilities may be history dependent, the SPCS is either {A,B,D,E}, {A,D,E}, {B,D,E} or {D,E}.

Summarizing, for Example 1.1', we have LCS = LCCS = vNM FSS = RE solutions = {D,E}, the ESS is either {A,B,D,E} or {D,E}, and SPCS is either {A,B,D,E}, {A,D,E}, {B,D,E} or {D,E}. Thus, all farsighted solution sets are also a SPCS. In particular, all farsighted solution concepts, except for the ESS and the SPCS, fail to identify the farsighted optimal behavior of Player 3 at State C, and predict a move from States A and B.

We conclude by another type of possible comparison between solution concepts. Say that solution concept $\mathcal{A}$ is weakly included in solution concept $\mathcal{B}$ if for any game, the union of all solution-$\mathcal{A}$ sets is a subset of the union of all solution-$\mathcal{B}$ sets. Accordingly, in the farsighted PD, any of the other farsighted solution concepts is not weakly included in the SPCS, while in Example 1.1', the SPCS is not weakly included in any of the other farsighted solution concepts. We conclude:

Corollary 4.1 *The following non-inclusion results hold: (i) Being a* SPCS *neither implies nor is implied by being any of the other farsighted solution concepts sets,* LCS*,* LCCS*,* vNM FSS*,* REFS*,* SREFS*,* HREFS*,* HSREFS*,* ESS *and* (C)ECB*.*

(ii) The SPCS *neither weakly includes nor is weakly included in any of the other farsighted solution concepts,* LCS*,* LCCS*,* vNM FSS*,* REFS*,* SREFS*,* HREFS*,* HSREFS*,* ESS *and* (C)ECB*.*

5 Summary

In this paper we introduce a new approach to farsightedness, embodied in the SPCS. The SPCS retains a main feature of several existing farsighted solution concepts, such as the LCS, LCCS, vNM FSS, and the more recently introduced solution concepts by D&V (2017), D&V(2020), and K&R(2021), namely a vNM type of consistency. However, as it relies on subgame perfection, the problem of maximality, raised by Ray and Vohra (2015a, 2019), is not present therein. Coalitions are employing best response choices and share the same beliefs on coalitional moves at each state. Further, according to the SPCS, all reachable stable states after a move from a current state are considered, and the attractiveness of such a move is determined by using preferences respecting first order stochastic dominance over the utilities of the reachable states. Thus, the SPCS incorporates inherent uncertainties in the model and as such, it extends the farsighted reasoning beyond the confidence assumption, which is integral in the solution concepts based on a rational expectation function (or coalition behavior) approach, recently introduced by D&V (2017), D&V (2020), K&R (2021) and Kimya (2020).

We prove that whenever the set of states, Z, is finite, there exists a SPCS*, and we further prove that the SPCS/SPCS* leads, e.g., to (weak) Pareto efficiency, without coalitions, in any normal form game having a myopic equilibrium. This result is shown to imply that farsighted players who adopt the SPCS/SPCS* reasoning will achieve full cooperation and overcome misaligned incentives in a variety of settings. Specifically, our farsighted players will always share the monopolistic profit in farsighted settings based on Bertrand and Cournot competition, and will always achieve supply chain coordination and Pareto efficiency in a decentralized setting of the classical newsvendor model and its variants. Finally, modeling network formation as 1-farsighted normal form game, we show, similar to, e.g., Kimya (2020) and Luo et al. (2021), that the SPCS/SPCS* is able to mollify the tension between stability and efficiency in network formation.

Appendix: Proofs

Proof of Proposition 2.1. The definition of $\Psi_{\sigma|h}$ implies that it is non-empty since the strategy profile $\sigma \in \Sigma_h$, following h, generates a probability measure having support consisting only of infinite converging histories. Therefore, by definition 2.1, a SPCS or SPCS* is necessarily non-empty. Any terminal state is necessarily in any SPCS or SPCS*, as no coalition can alter a terminal state when it is selected as an initial state. ■

Proof of Theorem 2.1. Existence is established assuming that preferences for any coalition S are represented by the expectation of a coalition utility function u_S that is a continuous and monotonic aggregation of the utility functions u_i for all $i \in S$ from final states, e.g. the expected sum of coalition member utilities from final states. We start by showing that $R^*(z) \neq \emptyset$ for any state $z \in Z$. Let $m = \min_{i \in N, \hat{z} \in Z} u_i(\hat{z}) - 1$ be a strict lower bound for the utility of any player in any state. Following Flesch et al. (2010), define an auxiliary game G in the class $\mathcal{G}^+$ of positive recursive stochastic games with complete information as follows: their non-empty and finite set of players N is our set of all non-empty coalitions; their non-empty and finite set of states S is $Z \cup Z'$, where their set of non-absorbing states is our set of states Z, and their set of absorbing states is Z', a duplicate of Z having as member one state z' for each and every state $z \in Z$; for each state $t \in S$, their associated controlling player i_t is some arbitrary non-empty coalition S_z in our game that has the effectiveness to move at the state z associated with t, or some arbitrary non-empty coalition S_z if no coalition has the effectiveness to move at z; for each state $t \in S$, their associated non-empty and finite set of actions A_t is $\{z' \in Z'\} \cup \{z'' \in Z \mid z \to_{S_z} z''\}$ consisting of the 'absorb/stay' action $z' \in Z'$ corresponding to z, together with our set of states z'' to which coalition S_z has the effectiveness to move from the state z associated with t; for each pair of states $t, u \in S$ and action $a \in A_t$, their transition probability is deterministic, i.e. the transition probability given action a from state $z \in Z$ to state $z'' \in Z \cup Z'$ associated with t, u, respectively, is $p_z(a, z'') = 1$ for $z'' = a$ and 0 otherwise; for each player $i \in N$, state $t \in S$ and action $a \in A_t$, their payoff $r_t^i(a)$ for every non-absorbing state t is 0, and for every absorbing state t equals $u_S(z) \equiv F_S[(u_j(z) - m)_{j \in S}]$ for the coalition S associated with i and the state $z \in Z$ corresponding to the $z' \in Z'$ associated with t, where $F_S : \mathbb{R}_+^S \to \mathbb{R}$ is a continuous and monotonic function; their initial state $s \in S$ is some initial state $z \in Z$ in our game. Since all transitions in G are deterministic, by the Main Theorem of Flesch et al. (2010), G has a subgame perfect equilibrium in pure strategies. This subgame perfect equilibrium is absorbing, i.e. absorption occurs eventually with probability 1. Since all transitions are deterministic and the strategies are pure, the subgame perfect equilibrium involves a unique absorbing state corresponding to a state $\bar{z} \in Z$. Since any pure strategy best response for a player in G following any history implies in our game the corresponding pure strategy best response under the corresponding preferences for the corresponding coalition following the corresponding history, the strategy profile σ (with deterministic protocol σ_c) corresponding to the subgame perfect equilibrium in G is a subgame perfect equilibrium in our game, and supports sure reachability from z to $\bar{z}$. Therefore, $R^*(z) \neq \emptyset$ for any state $z \in Z$. Note that any state $\bar{z} \in R^*(z)$ for any z is by definition surely reachable from itself, i.e. $\bar{z} \in R^*(\bar{z})$.

Next, define the set X inductively: Let $X^0 = Y^0 = \emptyset$; for $k = 1, 2, \ldots, |Z|$, define the sets X^k, Y^k as follows: if there exists a non-empty subset W^k of $Z \setminus Y^{k-1}$ that is strongly connected according to the relation R^* (i.e., any two states $z, \bar{z} \in W^k$ are connected by a path of sure reachability, so that each state is surely reachable from its predecessor along the path) and no state $\bar{z} \in Z \setminus (Y^{k-1} \cup W^k)$ is in $R^*(z)$ for some $z \in W^k$, then let $Y^k = Y^{k-1} \cup W^k$; additionally, define a set $V^k \subseteq W^k$ inductively: start with $V^{k,0} = W^k$, end with $V^k = V^{k,|W^k|}$, and for $l = 1, 2, \ldots, |W^k|$ let $V^{k,l} = V^{k,l-1} \setminus \{z^{k,l}\}$ if there exists $z^{k,l} \in V^{k,l-1}$ and a coalition S with the effectiveness to move at $z^{k,l}$ and a state $\bar{z}$ for which $z^{k,l} \to_S \bar{z}$ such that all $\hat{z} \in (X^{k-1} \cup V^{k.l-1}) \cap R^*(\bar{z})$ are weakly better for S than $z^{k,l}$ with at least one being strictly better, otherwise let $V^{k,l} = V^{k,l-1}$; then let $X^k = X^{k-1} \cup V^k$; in any other case, let

$Y^k = Y^{k-1}$ and $X^k = X^{k-1}$; finally, let $Y = Y^{|Z|}$ and $X = X^{|Z|}$.

By construction, each k, V^k and $z \in W^k$ satisfy the following two conditions, namely, condition (I): $z \in V^k$ if either no coalition has the effectiveness to move at z or there exists a coalition S^z that has the effectiveness to move at z and for any state $\bar{z}$ for which $z \to_{S^z} \bar{z}$ there exists $\hat{z} \in (X^{k-1} \cup V^k) \cap R^*(\bar{z})$ strictly worse for S^z than z, and condition (II): $z \notin V^k$ if there exists a coalition S^z with the effectiveness to move at z and a state $\bar{z}$ for which $z \to_{S^z} \bar{z}$ such that all $\hat{z} \in (X^{k-1} \cup V^k) \cap R^*(\bar{z})$ are weakly better for S^z than z with at least one being strictly better. Additionally, any state in Y is surely reachable from itself, while this does not hold for any state in $Z \setminus Y$. Observe also that $X \cap R^*(z) \neq \emptyset$ for any state $z \in Z$. To see this, note that (i) $z \in X$ implies $z \in R^*(z)$; (ii) for $z \in Y \setminus X$, condition (I) implies that for any coalition S^z that has the effectiveness to move at z and any state $\bar{z}$ for which $z \to_{S^z} \bar{z}$, any $\hat{z} \in (X^{k-1} \cup V^k) \cap R^*(\bar{z})$ is weakly better for S^z than z; and (iii) for $z \in Z \setminus Y$, $X \cap R^*(z) \neq \emptyset$ is supported by a strategy profile σ constructed with protocol σ_c always selecting for sure some single non-empty coalition that has the effectiveness to move from z, and applying backwards induction to extend the continuation strategy profiles that support the sure reachability of some state in $X \cap R^*(\bar{z})$ from each $\bar{z} \in Y$.

Now, define σ^{*X} according to the following four specifications, and we will subsequently show that it supports X, constructed above, as a SPCS:

Specification (1): after any history $h = (z^1, S^1, z^1, S^2, \ldots, z^1, S^t)$ such that $z^1 \in R^*(z^1)$ and $\{S^l\}_{l=1}^{t} \supseteq 2^N \setminus \emptyset$, i.e. with no moves by all non-empty coalitions from some initial state z^1 surely reachable from itself, let σ^{*X} continue according to some subgame perfect equilibrium σ supporting this sure reachability, i.e., for all h' that are continuation histories of h, $\sigma^{*X}_S(h') = \sigma_S(h')$ for all coalitions S with $P(h') = S$ and $\sigma^{*X}_c(h') = \sigma_c(h')$ when $P(h') = c$.

Specification (2): after any history $h = (z^1, S^1, z^1, S^2, \ldots, z^1, S^{t-1}, z^1)$ such that $\{S^l\}_{l=1}^{t-1} \not\supseteq 2^N \setminus \emptyset$, i.e. with no moves by some, but not all, non-empty coalitions from some initial state z^1, let the protocol $\sigma^{*X}_c(h)$ at this history have in its support only some arbitrary single coalition S^{z^1} that, when $z^1 \in Y$, satisfies the if statement in either condition (I) or (II) for z^1 in the role of z and some k, or, when the if statements in both conditions are violated or when $z^1 \in Z \setminus Y$, just has the effectiveness to move at z^1 if one exists, or any S^{z^1} otherwise.

Specification (3): after any history $h = (z^1, S^1, z^1, S^2, \ldots, z^1, S^t, z^2)$ such that $\{S^l\}_{l=1}^{t-1} \not\supseteq 2^N \setminus \emptyset$ and $z^1 \neq z^2$, i.e. with no moves from some initial state z^1 by some, but not all, non-empty coalitions, followed by a single, initial move by some coalition S^t to some other state z^2, consider some finite collection $C(z^1, S^t, z^2)$ of subgame perfect equilibria σ of a subgame in which z^2 is the initial state such that $\sigma_c(z^2) \neq \emptyset$, i.e. the first selected coalition in this subgame according to each $\sigma \in C(z^1, S^t, z^2)$ is non-empty, and where $C(z^1, S^t, z^2)$ satisfies that $z \in X \cap R^*(z^2)$ if and only if there is a corresponding subgame perfect equilibrium $\sigma^z \in C(z^1, S^t, z^2)$ having z as the sure final state; let the protocol $\sigma_c^{*X}(h')$ at any history $h' = (z^1, S^1, z^1, S^2, \ldots, z^1, S^t, z^2, \emptyset, z^2, \ldots, \emptyset, z^2)$, possibly with $\emptyset$ occurring $\tau = 0$ times, have for $S^{t+\tau+1}$ the support $\{\emptyset\} \cup [\cup_{\sigma^z \in C(z^1,S,z^2)} \sigma_c^z(z^2)]$, i.e. includes only the empty coalition and the first selected non-empty coalition in each $\sigma^z \in C(z^1, S, z^2)$; following any history $(z^1, S^1, z^1, S^2, \ldots, z^1, S^t, z^2, \emptyset, z^2, \ldots, \emptyset, z^2, S^{t+\tau+1})$ with $S^{t+\tau+1} \neq \emptyset$, let σ^{*X} continue according to each $\sigma^z \in C(z^1, S^t, z^2)$, the identity of which is determined by the identity of $S^{t+\tau+1}$ and, in case two distinct σ^z have the same first selected coalition, arbitrarily by τ, namely half the history length following z^2 until $S^{t+\tau+1}$, and if $S^{t+\tau+1}$ is different from the first selected coalition of any $\sigma^z \in C(z^1, S^t, z^2)$ then σ^{*X} continues according to an arbitrary $\sigma^z \in C(z^1, S^t, z^2)$ following the selection of $S^{t+\tau+1}$; more specifically regarding the protocol $\sigma_c^{*X}(h')$, let it assign probabilities to generate a distribution over σ^z, thus a distribution $d(z^2) \equiv d_{\sigma^{*X}|h}$ over final states z, so that, (i) whenever $z^1 \in V^k$ for some k and $S^t = S^{z^1}$ was used to justify this, the generated distribution $d(z^2)$ over final states following h makes S^{z^1} weakly prefer z^1 over $d(z^2)$ (this is possible either due to the existence in condition (I) of states in $(X^{k-1} \cup V^k) \cap R^*(z^2)$ strictly worse for S^{z^1} than z^1, with these final states being realized with sufficiently high probability and by preference continuity, or due to indifferences when the if statements in both conditions (I) and (II) are violated for z^1 in the role of z), and (ii) whenever $z^1 \in W^k \setminus V^k$ for some k and $S^t = S^{z^1}$ was used to justify this, the generated distribution $d(z^2)$ over final states following h makes S^{z^1} weakly prefer $d(z^2)$ over z^1 (this is possible either due to the existence condition (II) of states in $(X^{k-1} \cup V^k) \cap R^*(z^2)$ weakly better for S^{z^1} than z^1 with at least one being strictly better, with these final states being realized with sufficiently high probability and by preference continuity, or due to indifferences when the if statements in both conditions (I) and (II) are violated for z^1 in the role of z).

Specification (4): after any history $h = (z^1, S^1, z^1, S^2, \ldots, z^1, S^t)$ such that $\{S^l\}_{l=1}^{t-1} \not\supseteq 2^N \setminus \emptyset$, i.e. with no moves from some initial state z^1 by some, but not all, non-empty coalitions, followed by the selection of some coalition S^t, taking as given the continuation σ^{*X} as specified in Specification (3), coalition S^t acts optimally on and off play path when choosing whether or not to move from z^1 to some z^2.

Note that σ^{*X} is composed of subgame perfect equilibria following any history as in Specifications (1)-(3), and since each coalition only cares about final states, also following any history as in Specification (4). Consequently, σ^{*X} is a subgame perfect equilibrium. Furthermore, Specification (1) ensures that σ^{*X} satisfies the sure reachability $z \in R^*(z)$ according to requirement (a) in Definition 2.1. To see that it also satisfies the sure reachability for states in $X \cap R^*(z^2)$ according to requirement (b) and the stability of states in X according to requirement (c), note that, (i) for any initial state $z^1 \in X$ since $z^1 \in V^k$ for some k, coalition S^{z^1} that was used to justify this (and is selected for sure on play path) stays at z^1 because it weakly prefers z^1, which would be the final state for sure if S^{z^1} stayed at z^1, over $d(z^2)$, for any z^2 for which $z^1 \rightarrow_{S^{z^1}} z^2$ and $X^k \cap R^*(z^2) \neq \emptyset$, and because a move to z^2 for which $X^k \cap R^*(z^2) = \emptyset$ (consequently $(X \setminus X^k) \cap R^*(z^2) \neq \emptyset$ since $X \cap R^*(z^2) \neq \emptyset$) cannot be optimal, as this would contradict $(Z \setminus Y^k) \cap R^*(z^1) = \emptyset$ as required by the construction of Y^k; (ii) for any initial state $z^1 \notin X$ such that $z^1 \in W^k \setminus V^k$ for some k, coalition S^{z^1} that was used to justify this (and is selected for sure on play path) moves from z^1 to some z^2 for which $z^1 \rightarrow_{S^{z^1}} z^2$ and $X^k \cap R^*(z^2) \neq \emptyset$ because S^{z^1} weakly prefers $d(z^2)$ over z^1, which would be the final state for sure if S^{z^1} stayed at z^1, and because, as above, a move to z^2 for which $X^k \cap R^*(z^2) = \emptyset$ cannot be optimal, and (iii) some coalition S moves from any initial state $z^1 \notin X$ such that $z^1 \in Z \setminus Y$ because any such state is not surely reachable from itself. Therefore X is a SPCS* supported by σ^{*X}. ■

The following notation is used in the proof of Theorems 3.1 and 3.2. For every two states z, z' and coalition S, $z_S z'$ denotes the state $z'' \in Z$ defined by $z''_i = z_i$ for all $i \in S$ and $z''_i = z'_i$ otherwise.

Proof of Theorem 3.1. Although we refer in the proof to SPCS with its reachability R, the entire argument applies also to SPCS* with its reachability R^*. We first prove (1). Let $X \subseteq Z$ be a SPCS, thus it is necessarily non-empty by Proposition 2.1, and consider some subgame perfect equilibrium strategy profile σ^X that supports X as a SPCS. We first show that any state $z^2 \in X$ is surely reachable from any state $z^1 \in Z$. Fix such z^1, z^2. For any finite history h such that $h_0 = z^1$, i.e., z^1 is the initial state, let l_h^0 be the minimal even, positive number l such that $(z^2)_{h_{l-1}}(h_{l-2}) = z^2$, or let $l_h^0 = 0$ if such l does not exist. For the subgame in which z^1 is selected as the initial state, consider the strategy profile σ' defined with full support protocol σ'_c, and defined for any finite history h such that $h_0 = z^1$ and $P(h) = S$ as follows: if $\rightarrow_S = \emptyset$ then $\sigma'_S(h) = h_{K_h-1}$; if $\rightarrow_S \neq \emptyset$ and $l_h^0 = 0$ then let $\sigma'_S(h) = (z^2)_S(h_{K_h-1})$, otherwise let $\sigma'_S(h) = \sigma_S^X[z^2, (h_k)_{k=l_h^0-1}^{K_h}]$. In this subgame, according to σ', each coalition S, when selected to make a choice whether to keep the current state or to move to a new state, moves by choosing the alternatives $a_i \in A_i$ corresponding to z^2 for each player $i \in S$ and then does not move anymore, except after the first time a coalition S' was selected which could move from the current state z to z^2, i.e., $z_i = z_i^2$ for all $i \notin S'$, and in fact moved to z', in which case the strategy profile σ' continues exactly as σ^X does following the selection of z^2 as an initial state and after an initial move by S' from z^2 to z'. Consequently the state z^2 is the final state for sure according to σ' following z^1 as the initial state. It is also the final state for sure according to σ^X following the selection of z^2 as an initial state, because in this case σ^X dictates no moves – see the definition of a SPCS. Moreover, since σ^X is a subgame perfect equilibrium, no coalition strictly prefers to deviate from it after any history, in particular after a history in which all coalitions stayed at z^2. Additionally, no coalition strictly prefers to deviate from σ' after a history involving a partial process of moves from z^1 to z^2, because all other coalitions are dictated by σ' to move to z^2, and deviation alone of this coalition would lead to exactly the same weakly inferior distributions over final states as in the case of deviation of this coalition from z^2 according to σ^X. Therefore σ' is also a subgame perfect equilibrium, proving the sure reachability of z^2 from z^1. Since $X \subseteq R(z^1)$ for any $z^1 \in Z$, it follows that X is a countable set, as requirement (b) in Definition 2.1 of the SPCS implies that $X = \Psi_{\sigma^X|h}$, where $\Psi_{\sigma^X|h}$ is a countable set.

Now suppose that there are at least two distinct states in X and a player i that is not indifferent between them. Since a SPCS is compact and players' utility functions are continuous, there is a worst state $\check{z} \in X$ for player i. Consider the history $h = (\check{z}, \{i\})$, i.e., $\check{z}$ is selected as an initial state and player i is selected to make a choice whether to keep $\check{z}$ or to move to a new state. According to σ^X, since $\check{z} \in X$ and X is a SPCS, player i is supposed to keep $\check{z}$, anticipating it as the final state for sure – see the definition of a SPCS. But note that in case of an initial move by player i, off the σ^X equilibrium path, from $\check{z}$ to a new state, by requirement (b) in Definition 2.1 of the SPCS, any final state must be in X, and, since all states in X are reachable following a move away from $\check{z}$, there is a positive probability that the final state according to σ^X will be strictly better than $\check{z}$ for player i. Therefore, player i can either choose $\check{z}$ for sure, or alternatively, can choose a first order stochastically dominating distribution, as $\check{z}$ is the worst state in X for i. Since the first order stochastically dominating distribution is strictly preferred, the choice of player i to keep $\check{z}$ violates the assumption that σ^X is a subgame perfect equilibrium. Since this argument holds for any subgame perfect equilibrium strategy profile σ^X, we derived a contradiction to the assumption that X is a SPCS. Therefore, since X is non-empty, it must consist of states that are all equivalent for all players, i.e. $X \approx \{z^*\}$ for some $z^* \in Z$.

Suppose now that z^* is not Pareto efficient. Then there exists $d \in Z \backslash X$ such that $u_i(d) > u_i(z^*)$ for each $i \in N$. We first show that d is surely reachable from itself. For the subgame in which d is selected as the initial state, consider the strategy profile σ' defined with full support protocol σ'_c, and such that $\sigma'_S(h) = d$ for any coalition S and any finite history $h = (d, S^1, d, S^2, \ldots, d, S)$, and $\sigma'_S(h) = \sigma^X_S(h)$ for any other history h. In this subgame, according to σ', each coalition S stays at d, except when some coalition has previously not done so, in which case the strategy profile σ' continues exactly as σ^X does. Therefore σ' leads on equilibrium path to the final state d for sure. Moreover, σ' inherits its subgame perfection from σ^X after any history in this subgame with some move away from d, and it is also subgame perfect following a history with no previous such moves because requirement (b) in Definition 2.1 of the SPCS applied to σ^X implies that an initial move away from d

will lead for sure to a final state equivalent to z^*, which is strictly worse than d for any coalition. Therefore d is reachable from itself. Furthermore, applying the same argument to σ^X, any coalition anticipates that coalitions selected later to make a choice can either stay at d, which, by requirement (a) in Definition 2.1 of the SPCS and since d is reachable from itself, would lead to d as the final state for sure, or to make an initial move away from d, which, by requirement (b) of the SPCS, would lead for sure to a final state equivalent to the strictly worse z^*. Therefore, no coalition will move away from d after it was selected as an initial state, contradicting the definition of a SPCS because $d \notin X$. Thus z^* must be Pareto efficient.

Next we prove (2). Suppose that $X \approx \{z^*\}$ for some Pareto efficient $z^* \in Z$ such that $u_i(z^*) \geq u_i(e)$ for some myopic equilibrium $e \in Z$ and all $i \in N$. Index X with $\{z^t\}_{t=1}^{|X|}$. For any finite history h, let l_h^0 be the minimal even, positive number l such that $h_{l-1} \neq \emptyset$ and there exists an even, positive number $l' < l$ such that $h_{l'} \neq h_0$, or let $l_h^0 = 0$ if such l does not exist; let $t_h = 1 + \frac{1}{2} l_h^0 \mod |X|$; let $l_h^1 = 1$ if there exists an even, positive number $l > l_h^0$ such that $(h_l)_i \neq z_i^{t_h}$ for $i \in h_{l-1}$ and $\rightarrow_{h_{l-1}} \neq \emptyset$, otherwise let $l_h^1 = 0$. Consider the strategy profile σ^X defined with full support protocol σ_c^X following any finite history h such that $P(h) = c$, and defined for any finite history h such that $P(h) = S$ as follows: if $\rightarrow_S = \emptyset$ then $\sigma_S^X(h) = h_{K_h - 1}$, otherwise: (a) if $h_0 \in X$ then $(a1)$ $\sigma_S^X(h) = h_0$ when $l_h^0 = 0$, and $(a2)$ $\sigma_S^X(h) = (z^{t_h})_S(h_{K_h-1})$ when $l_h^0 > 0$ and $l_h^1 = 0$, and $(a3)$ $\sigma_S^X(h) = e_S(h_{K_h-1})$ when $l_h^1 = 1$; and (b) if $h_0 \notin X$ then $(b1)$ $\sigma_S^X(h) = h_0$ when $l_h^0 = 0$, $h_0 \in R(h_0)$ and $\{h_l\}_{l=1,3,\ldots,K_h} \supseteq \{S \subseteq N \mid \rightarrow_S \neq \emptyset\}$, otherwise $(b2)$ $\sigma_S^X(h) = (z^{t_h})_S(h_{K_h-1})$ when $l_h^1 = 0$, and $(b3)$ $\sigma_S^X(h) = e_S(h_{K_h-1})$ when $l_h^1 = 1$. According to σ^X, if an initial state in X is selected or if all effective coalitions were selected and none moved from an initial state reachable from itself, then no coalition moves and this is the final state for sure; otherwise, if an initial move is made away from an initial state in X, or if all previously selected coalitions stayed at an initial state not in X, then each $z^{t_h} \in X$, the identity of which is determined by the realization of the history length until the first selection of a non-empty coalition following the initial move, becomes a final state by each coalition moving by choosing the alternative corresponding to z^{t_h} and then not moving anymore; after any deviation from the above, each coalition moves to the alternative corresponding to e and then does not move anymore. This strategy profile clearly satisfies both requirements (a) and (b) in Definition 2.1 (note that since X is countable, there is no problem with the condition $\Psi_{\sigma^X|h} = X \cap R(z^2)$) and also requirement (c). We now verify that it forms a subgame perfect equilibrium. First note that on equilibrium path, whether or not the initial state is in X, any final state is equivalent for all players to z^*, and this holds also after an initial move away from an initial state in X. Any deviation after an initial move leads to the final state e for sure, and since $u_i(z^*) \geq u_i(e)$

for all $i \in N$, no coalition prefers to deviate from the equilibrium path, leading indeed to a final state in X. Moreover, after choices by all effective coalitions to stay off equilibrium path at an initial state reachable from itself, it becomes the final state for sure, supported in equilibrium by its own reachability. Similarly, e is reachable from itself because it is a myopic equilibrium, thus when reaching e off equilibrium path, no coalition prefers to deviate, as this can only lead to a final state at most as good as e. Consequently the strategy profile is a subgame perfect equilibrium. ■

Proof of Theorem 3.2. Although we refer in the proof to SPCS with its reachability R, the entire argument applies also to SPCS* with its reachability R^*. Suppose first that $X \subseteq \bar{Z}$ is a SPCS supported by σ^X. By Proposition 2.1, X is non-empty. Suppose that $u_i(z') < u_i^w$ for some $z' \in X$ and some player $i \in N$. Consider the case where z' is selected as an initial state and player i is selected to make a choice whether to keep z' or to move to a new state. Since $z' \in X$, by the definition of a SPCS, σ^X dictates that player i will keep z', anticipating it as a final state. But this player could instead adopt a strategy $\hat{\sigma}_{\{i\}}$ in which he always changes the current state whenever such an opportunity arises, thus forcing a swinging final state, as the game is generic. Since the utility, u_i^w, of a swinging final state is strictly higher than $u_i(z')$, the choice of player i to keep z' violates the subgame perfect equilibrium, a contradiction. Therefore $u_i(z') \geq u_i^w$ for all $z \in X$ and $i \in N$. We can now use an argument similar to the one used in the proof of Theorem 3.1, modified only in the part proving uniqueness up to equivalence for all players, which would instead conclude that $X \cap Z = \{z^*\}$. Then using the remaining argument concerning Pareto efficiency, swinging can be eliminated from X when it is not Pareto efficient. Therefore we conclude that $X = \{z^*\}$ for some Pareto efficient $z^* \in \bar{Z}$ such that $u_i(z^*) \geq u_i^w$ for all $i \in N$.

For the other direction, suppose that $X = \{z^*\}$ for some Pareto efficient $z^* \in \bar{Z}$ such that $u_i(z^*) \geq u_i^w$ for all $i \in N$. We use an argument similar to the one used to prove (2) in Theorem 3.1. For any finite history h, let l_h^0 be the minimal even, positive number l such that $h_{l-1} \neq \emptyset$ and there exists an even, positive number $l' < l$ such that $h_{l'} \neq h_0$, or let $l_h^0 = 0$ if such l does not exist; let $l_h^1 = 1$ if there exists an even, positive number $l > l_h^0$ such that $(h_l)_i \neq z_i^*$ for $i \in h_{l-1}$ and $\rightarrow_{h_{l-1}} \neq \emptyset$, otherwise let $l_h^1 = 0$. Consider the strategy profile σ^X defined with full support protocol σ_c^X following any finite history h such that $P(h) = c$, and defined for any finite history h such that $P(h) = S$ as follows: if $\rightarrow_S = \emptyset$ then $\sigma_S^X(h) = h_{K_h-1}$, otherwise: (a) if $h_0 \in X \cap Z$ then $(a1)$ $\sigma_S^X(h) = h_0$ when $l_h^0 = 0$, and $(a2)$ $\sigma_S^X(h) = (z^*)_S(h_{K_h-1})$ when $l_h^0 > 0$, $l_h^1 = 0$ and $z^* \neq w$, and $(a3)$ $\sigma_S^X(h) = z_S(h_{K_h-1})$ when $l_h^1 = 1$ or $z^* = w$, for some $z \in Z$ such that $z_S(h_{K_h-1}) \neq h_{K_h-1}$; and (b) if $h_0 \notin X$ then $(b1)$ $\sigma_S^X(h) = h_0$ when $l_h^0 = 0$, $h_0 \in R(h_0) \cap Z$ and $\{h_l\}_{l=1,3,\ldots,K_h} \supseteq \{S \subseteq N \mid \rightarrow_S \neq \emptyset\}$, otherwise $(b2)$ $\sigma_S^X(h) = (z^*)_S(h_{K_h-1})$ when $l_h^1 = 0$ and $z^* \neq w$, and $(b3)$ $\sigma_S^X(h) = z_S(h_{K_h-1})$

when $l_h^1 = 1$ or $z^* = w$, for some $z \in Z$ such that $z_S(h_{K_h-1}) \neq h_{K_h-1}$. According to σ^X, if an initial state in $X \cap Z$ is selected or if all effective coalitions were selected and none moved from an initial state reachable from itself, then no coalition moves and this is the final state for sure; otherwise, if an initial move is made away from an initial state in $X \cap Z$, or if all previously selected coalitions stayed at an initial state not in $X \cap Z$, then z^* becomes the final state for sure by each coalition moving by choosing the alternatives corresponding to z^* and then not moving anymore, or by keep changing the current state whenever it is selected to make a choice in the case of swinging; after any deviation from the above, each coalition conducts a swinging behavior. This strategy profile clearly satisfies both properties (a)-(c) in Definition 2.1. We now verify that it forms a subgame perfect equilibrium. First note that on equilibrium path, whether or not z^* is the initial state, z^* is the final state for sure, and this is also the final state for sure after an initial move away from the initial state z^*. Any deviation after an initial move leads to a swinging final state, in which case each player has utility u_i^w, as the game is generic, and since $u_i(z^*) \geq u_i^w$ for all $i \in N$, no coalition prefers to deviate from the equilibrium path, leading indeed to the final state z^* for sure. Moreover, after choices by all effective coalitions to stay off equilibrium path at an initial state reachable from itself, it becomes the final state for sure, supported in equilibrium by its own reachability. Similarly, a swinging final state is reachable from itself because it is forced by the other coalitions when they are selected to make a choice. Consequently the strategy profile is a subgame perfect equilibrium. ■

Proof of Lemma 4.1. Recall that a rational extended expectation function REEF, denoted $F = (F^1(z), \ldots, F^{k(z)}(z))$, where $F^j(z) = (f^j(z), S^j(z))$, $j = 1, \ldots, k(z)$, $S^{k(z)} = \emptyset$, $f^{k(z)}(z) = z$, is required to satisfy Conditions (I), (E), and (M). Now, for the PD game, we first show that $\{(\text{C,C}), (\text{D,D})\}$ is an ESS by the following REEF F: $F(\text{C,D}) = [((\text{D,D}), \{1\}), ((\text{C,D}), \emptyset)]$, $F(\text{D,C}) = [((\text{D,D}), \{2\}), ((\text{D,C}), \emptyset)]$, $F(\text{C,C}) = [((\text{C,C}), \emptyset)]$ and $F(\text{D,D}) = [((\text{D,D}), \emptyset)]$. To see this, consider each $z \in Z$, starting from $z = (\text{C,D})$. Condition I is satisfied since for coalition $T = \{2\} \notin \{\{1\}, \emptyset\}$, necessarily $y = (\text{C,C})$ with $z \to_{\{2\}} y$ and $i = 2 \in T$, and $l = k(z) = 2$, we have that $u_i(f^l(z), F) = 4 \geq 3 = u_i(y, F)$. Condition E is satisfied since $u_1(f^1(\text{C,D}), F) = 1 > 0 = u_i(f^2(\text{C,D}), F)$, and Condition M is satsified vacuously since there is no $y \in Z$ with $y \neq f^1(z)$ and $z \to_{S^1(z)} y$. Similarly, Conditions (I),(E) and (M) are satisfied for $z = (\text{D,C})$. Now consider $z = (\text{D,D})$. Condition (I) is satisfied since for coalition $T = \{1\} \notin \{\emptyset\}$, necessarily $y = (\text{C,D})$ with $z \to_{\{1\}} y$ and $i = 1 \in T$, and $l = k(z) = 1$, we have that $u_i(f^l(z), F) = 1 \geq 1 = u_i(y, F)$, and similarly for coalition $T = \{2\}$. Conditions (E) and (M) are satisified vacuously for $z = (\text{D,D})$. Finally, for $z = (\text{C,C})$, Condition (I) is satisfied since for coalition $T = \{1\} \notin \{\emptyset\}$, necessarily $y = (\text{D,C})$ with $z \to_{\{1\}} y$ and $i = 1 \in T$, and $l = k(z) = 1$, we have that

$u_i(f^l(z), F) = 3 \geq 1 = u_i(y, F)$, similarly for coalition $T = \{2\}$, and Conditions (E) and (M) are satisfied vacuously. Note that F is the unique REEF supporting $\{(\text{C,C}), (\text{D,D})\}$ as an ESS. To see this, suppose that $F(\text{C,D}) = [((\text{C,C}), \{2\}), ((\text{D,D}), \{1\}), ((\text{C,D}), \emptyset)]$. Then, for $z = (\text{D,D})$, $T = \{1\} \notin \{\emptyset\}$, $f^l(z) = z$, necessarily $y = (\text{C,D})$ with $z \rightarrow_{\{1\}} y$ and $i = 1 \in T$, we have $u_i(f^l(z), F) = 1 \ngeq 3 = u_i(y, F)$, which contradicts Condition (I). Any other possibility for $F(\text{C,D})$ involves coalition $\{1, 2\}$, but since this coalition does not have any effectiveness to move in this game, any such possibility violates Condition (E) at State (C,D). Similar reasoning apply for State (D,C).

Next we show $\{(\text{C,C}), (\text{D,D})\}$ is the unique ESS. To this end, consider any other ESS. For each of the following mutually exclusive and exhaustive cases for this ESS we show a contradiction, which implies that no other ESS exists.

Case 1: State (D,D) is in the ESS, i.e. it is stationary according to F. Then State $z = (\text{C,D})$ is not in the ESS, since otherwise it is stationary and for $k(z) = 1 = l$, $S^1(z) = \emptyset$, $T = \{1\} \notin \{S^1(z)\}$, $f^l(z) = z$ and necessarily $y = (\text{D,D})$ with $z \rightarrow_{\{1\}} y$ and $i = 1 \in T$, we have $u_i(f^l(z), F) = 0 \ngeq 1 = u_i(y, F)$, which contradicts Condition (I). Therefore $k(z) > 1$, and for $l = k(z) - 1$ it must be that $f^l(z) = (\text{D,D})$ and $S^l(z) = \{1\}$, which is the only way to satisfy Condition (E) by $u_i(f^l(z), F) = 1 > 0 = u_i(f^{l+1}(z), F)$ for all $i \in S^l(z)$. Similarly, (D,C) is not in the ESS and $k(\text{D,C}) > 1$. Then, State $z = (\text{C,C})$ is stationary, since otherwise (D,D) is the unique stationary state and for $l = k(z) - 1$ it must be that $u_i(f^l(z), F) = 1 \ngtr 3 = u_i(f^{l+1}(z), F)$ for $i \in S^l(z)$, which contradicts Condition (E). Since the considered ESS was assumed different from $\{(\text{C,C}), (\text{D,D})\}$, this is a contradiction.

Case 2: State (D,D) is not in the ESS, and State (C,C) is in the ESS. Then, State (C,D) is not in the ESS, since otherwise for $z = (\text{C,C})$, $T = \{2\} \notin \{\emptyset\}$, $f^l(z) = z$, necessarily $y = (\text{C,D})$ with $z \rightarrow_{\{2\}} y$ and $i = 2 \in T$, we have $u_i(f^l(z), F) = 3 \ngeq 4 = u_i(y, F)$, which contradicts Condition (I). Similarly, State (D,C) is not in the ESS. Since (C,C) is the unique stationary state, for $z = (\text{C,D})$, $l = k(z) - 1$, $f^l(z) = (\text{D,D})$ and $S^l(z) = \{1\}$ we have $u_1(f^l(z), F) = 3 > 0 = u_1(f^{l+1}(z), F)$ in order to satisfy Condition (E). But then, for $l = 1$ we necessarily have $f^l(z) = (\text{C,C})$, $S^l(z) = \{2\}$ and $u_1(f^l(z), F) = 3 \ngtr 3 = u_1(f^{l+1}(z), F)$, which contradicts Condition (E).

Case 3: Neither (D,D) nor (C,C) is in the ESS. By Corollary 4.1 in K&R (2021), the ESS must be non-empty. If both (C,D) and (D,C) are in the ESS alone then for $z = (\text{D,D})$ and $l = k(z) - 1$ we have $u_i(f^l(z), F) = 0 \ngtr 1 = u_i(f^{l+1}(z), F)$ for $i \in S^l(z)$, which contradicts Condition (E). If (C,D) is the unique state in the ESS then we must have that $F(\text{D,D}) = [((\text{D,C}), \{2\}), ((\text{D,D}), \emptyset)]$ so that $u_2(f^1(z), F) = 4 > 1 = u_2(f^2(z), F)$ in order to satisfy Condition (E). However, for $z = (\text{D,C})$ and $l = k(z) - 1$ necessarily $f^l(z) = (\text{C,C})$, $S^l(z) = \{1\}$ and $u_1(f^l(z), F) = 0 \ngtr 4 = u_1(f^{l+1}(z), F)$, which contradicts Condition (E).

Similarly, (D,C) cannot be the unique state in the ESS.

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References

Bala, V., S. Goyal. 2000. A noncooperative model of network formation. Econometrica, 68(5), 1181-1229.

Bloch, F., A. van den Nouweland. 2020. Farsighted stability with heterogeneous expectations, Games and Economic Behavior, 121, 32-54.

Bloch, F., A. van den Nouweland. 2021. Myopic and farsighted stable sets in 2-player strategic-form games. Games and Economic Behavior, 130, 663-683.

Brams, S.J. 1994. Theory of Moves. Cambridge University Press, New York, NY.

Brams, S.J., D. Wittman. 1981. Nonmyopic Equilibria in 2x2 Games. Conflict Management and Peace Science, 6, 39-62.

Cachon, G.P., M.A. Lariviere. 2005. Supply Chain Coordination with Revenue-Sharing Contracts: Strengths and Limitations, Management Science, 51 (1), 30-44.

Cai, X., M. Kimya. 2023. Stability of Alliance Networks. Games and Economic Behavior, 140, 401-409.

Chamberlin, E.H. 1933. The theory of monopolistic competition. Harvard University Press, Cambridge, MA.

Chwe, M.S-Y. 1994. Farsighted Coalitional Stability. Journal of Economic Theory, 63, 299-325.

Diamantoudi, E., L. Xue. 2003. Farsighted Stability in Hedonic Games. Social Choice and Welfare, 21(1), 39-61.

Dutta, B., S. Ghosal, D. Ray. 2005. Farsighted network formation. Journal of Economic Theory, 122(2), 143-164.

Dutta, B., H. Vartiainen. 2020. Coalition formation and history dependence. Theoretical Economics, 15(1), 159-197.

Dutta, B., R. Vohra. 2017. Rational expectations and farsighted stability. Theoretical Economics, 12(3), 1191-1227.

Flesch, J., J. Kuipers, G. Schoenmakers, K. Vrieze. 2010. Subgame Perfection in Positive Recursive Games with Perfect Information. Mathematics of Operations Research, 35(1), 193-207.

Granot, D., G. Sošić. 2005. Formation of Alliances in Internet Based Supply Exchanges. Management Science 51 (1), 92-105.

Granot, D., S. Yin. 2008. Competition and Cooperation in Decentralized Push and Pull Assembly Systems. Management Science, 54 (4), 733-747.

Greenberg, J.H. 1990. The Theory of Social Situations. Cambridge University Press, New York, NY.

Harsanyi, J.C. 1974. An Equilibrium-Point Interpretation of Stable Sets and a Proposed Alternative Definition. Management Science, 20 (11), 1472-1495.

Herings, P.J.J., A. Mauleon, V. Vannetelbosch. 2004. Rationalizability for social environments. Games and Economic Behavior, 49(1), 135-156.

Herings, P.J.J., A. Mauleon, V. Vannetelbosch. 2009. Farsightedly stable networks. Games and Economic Behavior, 67(2), 526–541.

Hirai, T., N. Watanabe, S. Muto. 2019. Farsighted stability in patent licensing: An abstract game approach. Games and Economic Behavior, 118, 141-160.

Jackson, M.O. 2010. Social and economic networks. Princeton university press.

Jackson, M.O., A. Wolinsky. 1996. A strategic model of social and economic networks. Journal of economic theory, 71(1), 44-74.

Jordan, J.S. 2006. Pillage and property. Journal of economic theory, 131(1), 26-44.

Karos, D., L. Robles. 2021. Full farsighted rationality. Games and Economic Behavior, 130, 409-424.

Kawasaki, R. 2015. Maximin, minimax, and von Neumann–Morgenstern farsighted stable sets. Mathematical Social Sciences, 74, 8–12.

Kawasaki, R., S. Muto. 2009. Farsighted stability in provision of perfectly "Lumpy" public goods. Mathematical Social Sciences, 58, 98-109.

Kilgour, D.M. 1984. Equilibria for far-sighted players. Theory and Decision, 16, 135-157.

Kimya, M. 2020. Equilibrium coalitional behavior. Theoretical Economics, 15(2), 669-714.

Konishi, H., D. Ray. 2003. Coalition formation as a dynamic process. Journal of Economic Theory, 110, 1–41.

Krishnan, H., R.A. Winter. 2011. On the Role of Revenue-Sharing Contracts in Supply Chains. Operations Research Letters, 39 (1), 28-31.

Luo, C., A. Mauleon, V. Vannetelbosch. 2021. Network formation with myopic and farsighted players. Economic Theory, 71(4), 1283-1317.

Mariotti, M. 1997. A Model of Agreements in Strategic Form Games. Journal of Economic Theory, 74, 196-217.

Masuda, T., A. Suzuki, S. Muto. 2000. Farsighted von Neumann-Morgenstern Stability Leads to Efficiency in Oligopoly Markets. Working paper.

Mauleon, A., V. Vannetelbosch. 2004. Farsightedness and Cautiousness in Coalition Formation Games with Positive Spillovers. Theory and Decision, 56, 291–324.

Mauleon, A., V. Vannetelbosch, W. Vergote. 2011. von Neumann–Morgenstern Farsightedly Stable Sets in Two-Sided Matching. Theoretical Economics 6, 499–521.

Muto, S. 1993. Alternating-move preplays and vNM stable sets in two person strategic form games. Tilburg CentER working paper series 9371.

Nagarajan, M., Y. Bassok. 2008. A Bargaining Framework in Supply Chains: The Assembly Problem. Management science, 54 (8), 1482-1496.

Nagarajan, M., G. Sošić. 2007. Stable Farsighted Coalitions in Competitive Markets. Management Science, 53 (1), 29-45.

Nagarajan, M., G. Sošić. 2009. Coalition Stability in Assembly Models. Operations Research, 57 (1), 131–145.

Nakanishi, N. 2009. Noncooperative farsighted stable set in an n-player prisoners' dilemma. International Journal of Game Theory, 38(2), 249-261.

Osborne, M.J., A. Rubinstein. 1994. A Course in Game Theory. MIT Press, Cambridge, MA.

Page, F.H., M.H. Wooders, S. Kamat. 2005. Networks and Farsighted Stability. Journal of Economic Theory, 120, 257–269.

Page, F.H., M.H. Wooders. 2009. Strategic basins of attraction, the path dominance core, and network formation games. Games and Economic Behavior, 66, 462–487.

Pearce, D.G. 1984. Rationalizable strategic behavior and the problem of perfection. Econometrica, 52 (4), 1029-1050.

Ray, D., R. Vohra. 2015a. Coalition formation. In Handbook of Game Theory, Vol. 4, ed. H.P. Young and S. Zamir. Amsterdam: North-Holland, 239–326.

Ray, D., R. Vohra. 2015b. The farsighted stable set. Econometrica, 83(3), 977-1011.

Ray, D., R. Vohra. 2019. Maximality in the farsighted stable set. Econometrica, 87(5), 1763-1779.

Sošić, G. 2006. Transshipment of Inventories Among Retailers: Myopic vs. Farsighted Stability. Management Science, 52 (10), 1493-1508.

Suzuki, A., S. Muto. 2005. Farsighted stability in an n-person prisoner's dilemma. International Journal of Game Theory, 33(3), 431-445.

Suzuki, A., S. Muto. 2006. Farsighted Behavior Leads to Efficiency in Duopoly Markets. Annals of the International Society of Dynamic Games, 8(6), 379-395.

von Neumann, J., O. Morgenstern. 1944. Theory of Games and Economic Behavior. New York: John Wiley and Sons.

Xue, L. 1998. Coalitional Stability under Perfect Foresight. Economic Theory, 11 (3), 603-627.